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Research Article

Curvature Dependent Electrostatic Field in the Deformable MEMS Device: Stability and Optimal Control

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Abstract

The recovery of the membrane profile of an electrostatic micro-electro-mechanical system (MEMS) device is an important issue because, when applying an external voltage, the membrane deforms with the consequent risk of touching the upper plate of the device (a condition that should be avoided). Then, during the deformation of the membrane, it is useful to know if this movement admits stable equilibrium configurations. In such a context, our present work analyze the behavior of an electrostatic 1D membrane MEMS device when an external electric voltage is applied. In particular, starting from a well-known second-order elliptical semi-linear differential model, obtained considering the electrostatic field inside the device proportional to the curvature of the membrane, the only possible equilibrium position is obtained, and its stability is analyzed. Moreover, considering that the membrane has an inertia in moving and taking into account that it must not touch the upper plate of the device, the range of possible values of the applied external voltage is obtained, which accounted for these two particular operating conditions. Finally, some calculations about the variation of potential energy have identified optimal control conditions.

Keywords: MEMS DEVICE, ELECTROSTATIC ACTUATORS, BOUNDARY SEMI-LINEAR ELLIPTC MODELS, CURVATURE

AMS subject classification: 34B15, 53A04, 74K15, 93D05

1. Introduction to the Problem

In the last decade, the evolution of engineering applications had a strong increase towards embedded technologies, in which the small size of the devices used played a fundamental role, representing the link between the physical nature of the problem and the logic of the machine language [1]. In this context, the interest regarding micro-electro-mechanical system (MEMS) devices has matured, as they are devices that effectively support the interface between man and electronics [1]. Historically, the birth of MEMS dates back to 1964 with the production of the first batch device [2]. MEMS technology has since been further developed, starting from engineering fields up to multi-physics-mathematical disciplines due to advanced theoretical models that allow the explicit obtaining of solutions [3,4]. Alternatively, if such solutions were not explicitly obtainable, it is desirable to obtain conditions that guarantee the existence, uniqueness, and regularity of the solutions [5,6]. Furthermore, if the explicit solution is not obtainable, it is possible to obtain solutions by means of numerical techniques, where, if they satisfy the analytical conditions of existence, uniqueness, and regularity, the absence of ghost solutions is ensured [7–10].

The scientific community, in the MEMS field, is busy on two main fronts. The first one, theoretically oriented, is devoted to the analysis and synthesis of physical-mathematical models [11] of problems, such as coupled thermal-elastic systems [9,11,12], electrostatic-elastic systems [11,13,14], magnetically actuated systems, and microfluidics [11,15–18]; the second front is actively engaged in technology transfer in various application areas, such as the production of MEMS for biomedical applications (miniaturized bio-sensors, tissue engineering, and so on) [14,19]. Clearly, there is no lack of lines of research that combine the theoretical approach with technology transfer [1,11].

This is the case of highly specialized research fields, such as the modeling of magneto-static-thermo-elastic devices, where the analytical models also allow the study of wave propagation in micro-domains with fixed or free boundaries [20–23]; or, obtaining the conditions of existence and uniqueness of solutions to inverse problems with moving boundaries, and many other interesting applications [5,24,25]. Due to the reduced maintenance required and the simple construction technology, membrane MEMS devices represent excellent candidates in many engineering applications [11,22,26–28]. Many of them admit a simple 1D representation; thus, our interest is focused on them.

With these premises, considering a closed region Ω, a 1D membrane MEMS device can be represented by two parallel plates with a mutual distance d. A membrane with the same dimensions of the plate is clumped on the edge of the lower plate and free to deform towards the upper plate when an external electric voltage V is applied between the plates [5,7]. For more details, see Section 3 and Fig. 1a.
As known, the dimensionless mathematical model for this device is:

$$\begin{cases} \frac{d^2u(x)}{dx^2} = -\frac{\lambda^2}{(1-u(x))^2} & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

in which $\lambda^2$is a parameter proportional to the external V. However, in (1), the quantity $\frac{\lambda^2}{(1-u(x))^2}$ is proportional to $|\mathbf{E}|^2$, where $\mathbf{E}$ is the electrostatic field inside the device. Thus, the following relationship makes sense [5,7,25]:

$$\frac{d^2u(x)}{dx^2} = -\theta|\mathbf{E}|^2 \quad \theta \in \mathbb{R}^+. \tag{2}$$

Moreover, as explicated in [5], $\mathbf{E}$ on the membrane can be locally considered orthogonal to the straight line tangent to the membrane, so that $|\mathbf{E}|$ can be computated as proportional to the curvature of the membrane

$$K(x, u(x)) = \left|\frac{d^2u(x)}{dx^2}\right| \left(\sqrt{\left(1 + \left(\frac{du(x)}{dx}\right)^2\right)^3}\right)^{-1}, \tag{3}$$

obtaining, after some mathematical steps, the following 1D second-order semi-linear elliptic model:

$$\begin{cases} \frac{d^2u(x)}{dx^2} = -\frac{1}{\theta\lambda^2}\left(1 + \left(\frac{du(x)}{dx}\right)^2\right)^3 (1 - u(x) - d^*)^2 \ \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega; \qquad 0 \le u(x) \le 1 - d^*, \end{cases} \tag{4}$$

in which d^* is the critical security distance ensuring that the membrane does not touch the upper plate of the device. Model (4) was studied extensively in [5], where the authors obtained a condition ensuring both the existence and uniqueness of the solution. Once this condition is achieved, two types of problems arise: the first type relates to problems inherent in stability, while the second relates to problems relating to the electrical voltage V applied. There is no doubt that the most dangerous configuration of the membrane is when the center point $u(0) = u_0$, reaches the maximum allowable height, that is $1 - d^*$.

In this work, we asked: is the configuration of the membrane, for which u_0 reaches the value $1 - d^*$, an equilibrium position for the membrane, and, if so, is this equilibrium position of the membrane also stable? At the same time, we considered that the external voltage, V, provides the electrostatic pressure, p_{el}, which determines the mechanical pressure, p, that is necessary for the movement of the membrane; we also sought what will be the minimum value of V necessary to overcome the inertia of the membrane and, moreover, which will be the maximum admissible value of V necessary to reach the configuration of the membrane in which u_0 assumes the maximum allowable height, $1 - d^*$ that cannot be exceeded. Finally we also asked if it were possible to obtain conditions starting from energy computations that ensure optimal control of the device.

The paper is structured as follows. After a brief overview of the scientific literature related to the problem under study (Section 2), the 1D membrane MEMS devices is described in detail in Section 3. In Section 4, we explain how the equilibrium position was achieved. In particular, starting from the transformation of the second-order differential model in a suitable first-order differential system (Subsection 4.1) the equilibrium configuration was obtained using an established technique (Subsection 4.2). Then, the stability of the equilibrium configuration was studied in Section 5, exploiting an approach based on the linearization of the first-order differential system around the equilibrium configuration (Subsection 5.1) for tracing the orbits to exploit Lyapunov's theory.

Once the stability of the single point of equilibrium was studied, Section 6 is entirely dedicated to the study of the admissible values for V to ensure proper device operation. In particular, Subsection 6.2 deals with the study of the values of V necessary to overcome the inertia of the membrane in the start-up phase while the following Subsection 6.3 evaluates the value of V necessary to reach the maximum allowed height of the membrane. Finally, some conclusions with ideas for future works are reported in Section 8.

2. 1D Electrostatic Membrane MEMS: a Brief Scientific Literature Review

As known, the starting point of the 1D electrostatic MEMS device is the following model

$$\begin{cases} \alpha\Delta^2 u(x) = \left(\varrho\int_\Omega |\nabla u(x)|^2 dx + \gamma\right)\Delta u(x) + \dfrac{\lambda_1 f_1(x)}{(1-u(x))^\sigma\left(1+\chi\int_\Omega \frac{dx}{(1-u)^{\sigma-1}}\right)} \\ u(x) = \Delta u(x) - du_\nu = 0, \quad x\in\partial\Omega, \;\; d\geq 0; \qquad 0\leq u(x)<1, \quad x\in\Omega, \end{cases} \tag{5}$$

studied in [29]. This model exploits the Steklov boundary condition to obtain Dirchlet and Navier boundary conditions, related to a MEMS device constituted by means of two plates. The upper plate is undeformable while the lower one (also called the ground plate) is deformable when an external V is applied. As in (5), f_1 is a bounded function taking into account the dielectric properties of the material constituting the deformable plate; λ_1 is the external applied voltage; $\alpha, \varrho, \gamma, \chi$ are positive parameters related to the electric and mechanic properties of the material constituting the deformable plate; $\sigma \geq 2$ takes into account the more general Coulomb's exponents. Model (5) is the generalization of the following dimensionless model

$$\begin{cases} \Delta^2 u(x) = \frac{\lambda_1 f_1(x)}{[1-u(x)]^2} \\ 0\leq u(x) < 1 \;\text{ in } \Omega, \qquad u = \Delta u - du_\nu, \;\text{ on } \partial\Omega, \;\; d\geq 0, \end{cases} \tag{6}$$

studied in [30], where the thickness of the ground plate is zero so that the inertial effects can be neglected as well as the non-local effects ($\sigma = 2$, $\alpha = 1$, $\varrho = \gamma = \chi = 0$). Moreover, model (6) has been exploited in [5] in the following dimensionless form where $\lambda^2 = \lambda_1 f_1(x)$:

$$\begin{cases} \frac{d^2u(x)}{dx^2} = -\frac{\lambda^2}{(1-u(x))^2} \;\text{ in } \Omega \\ u(-L) = u(L) = 0, \end{cases} \tag{7}$$

where L is the semi-length of the dimensionless 1D device, to propose a variant in which λ^2 is related to V by the following expression [11]:

$$\lambda^2 = \frac{\epsilon_0 V^2 (2L)^2}{2d^3 T}, \tag{8}$$

in which ϵ_0 is the permittivity of the free space, d is the distance between the two parallel plates of the device, and T is the mechanical tension of the membrane. From a physical point of view, λ^2 represents the ratio of a reference electrostatic force to a reference elastic force [5,11].

Remark 2.1. λ^2 is also linked to $|\mathbf{E}|$. In fact, considering both (1) and (2), we can write:

$$\theta|\mathbf{E}|^2 = \frac{\lambda^2}{(1-u(x))^2} = \frac{\epsilon_0 V^2 (2L)^2}{2d^3 T(1-u(x))^2}. \tag{9}$$

Multiplying (9) by λ^2 and taking into account (8), the following equality holds:

$$\theta\lambda^2 = \frac{\epsilon_0^2 L^4}{d^6 T^2} \frac{V^4}{(1-u(x))^2|\mathbf{E}|^2}. \tag{10}$$

Remark 2.2. We observe that from (10), if $|\mathbf{E}|$ increases, $\theta\lambda^2$ decreases so that, by the equation of the model (4), $\left|\frac{d^2u(x)}{dx^2}\right|$ also increases, and thus the concavity of the membrane increases as well.

As $\mathbf{E}$ on the membrane is physically locally orthogonal to the tangent straight line of the membrane, in [5] $|\mathbf{E}|$ was considered as proportional to the curvature $K(x, u(x))$ of the membrane obtaining the following model

$$\begin{cases} -\frac{d^2u(x)}{dx^2} = \theta\mu^2(x,u(x),\lambda)K^2(x,u(x)) = \theta\lambda^2 \frac{K^2(x,u(x))}{(1-u(x)-d^*)^2} \text{ in } \Omega = (-L, L) \\ u(-L) = u(L) = 0; \quad 0 \le u(x) \le 1-d^* \end{cases} \tag{11}$$

where $\mu(x, u(x), \lambda) = \frac{\lambda}{1-u(x)-d^*}$, in which $d^* = \frac{\lambda}{\epsilon_t}$ [11], where ϵ_t is the dielectric strength of the material constituting the membrane. Then, taking into account Equation (3), model (11), as described in [24], became the model (4), in which the singularity $1 - u(x)$ is not explicit. Model (4) has been studied in depth and a result of the existence of one solution has been achieved exploiting Green's functions and the Schauder–Tychonoff fixed point theory as well.

Remark 2.3. Concerning the uniqueness of the solution for (4), even though in [24] it was always ensured independently of the electromechanical properties of the material constituting the membrane, in [7] a new condition ensuring the uniqueness of the solution was achieved, depending on those properties. Then, in [7], a new condition ensuring both the existence and uniqueness of the solution for (4), depending on the electromechanical properties of the material constituting the membrane of the device was achieved. Finally, in [7], both the existence and uniqueness of the solution for (4) were exploited to numerically recover the profile of the membrane in the presence/absence of ghost solutions.

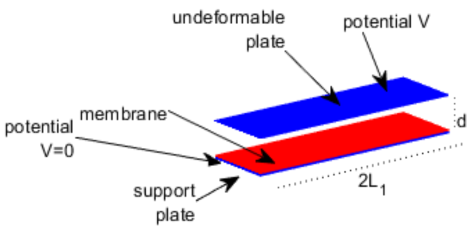
3. A Brief Description of the 1D Electrostatic Membrane MEMS Device

The device studied in this work consisted of two parallel metallic plates where the distance between them was d. If the lower plate is located on the x axes of a system of Cartesian axes Oxy, if $V = 0$, the membrane is not deformed and lies on the lower plate so that its profile, $u(x) = 0$; when $V > 0$ is applied, the membrane deforms and its profile $u(x) > 0$ while $u(x) = 0$ at $x = -L$ and $x = +L$; in fact on the lower plate, there was a membrane free to move but anchored to the edge of this plate, i.e., at $x = -L$ and $x = +L$. The upper plate and the membrane were assumed to be perfect conductors. We assumed that the length of the plates, L, was much larger than the distance d, that is $L \gg d$. Dimensionally, d is of the order of 10^{-9} while L is of the order of 10^{-6}. In such conditions, since $L \gg d$, it follows to consider the device in 1D geometry. Then, the device can be schematized as in Fig. 1a.

When an external V was applied, the membrane moves towards the upper plate. In particular, to overcome the mechanical inertia of the membrane, it was necessary to apply a minimum V, below which the membrane did not move. However, to move the membrane, the applied voltage V, in addition to being greater than the minimum value of V to overcome the mechanical inertia, must assume a value limited by a maximum value so that the membrane does not touch the upper plate, making the device unusable. The operating conditions whereby the membrane assumes the maximum possible deformation (i.e., very close to the upper plate) requires particular attention. In particular, we ask ourselves if this position of the membrane inside the device represents an equilibrium configuration and, moreover, if this equilibrium is stable.

4. Search of Equilibrium Configurations

To obtain the equilibrium configurations for the membrane whose profile is $u(x)$, we must transform model (4) into a corresponding system of two first-order differential equations in normal form, i.e., in the



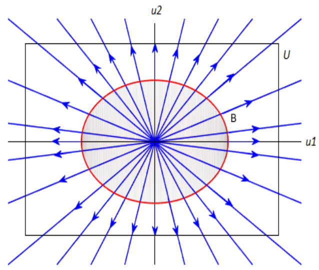


Figure 1. (a) The 1D membrane MEMS device. (b) Orbits for the system (26).

form:

$$\begin{cases} \frac{du_1(x)}{dx} = \overline{f}(u_1(x), u_2(x)) \\ \frac{du_2(x)}{dx} = \overline{g}(u_1(x), u_2(x)). \end{cases} \tag{12}$$

Afterward,it is sufficient to simply to vanish $\overline{f}(u_1(x), u_2(x))$ and $\overline{g}(u_1(x), u_2(x))$ [31].

4.1. *A Suitable Version of the 1D second-order semi-linear elliptic Model: an Equivalent Non-Linear first-order Differential Equation System*

To write model (4) as a system of first-order differential equations, we considered two auxiliary functions, $u_1(x)$ and $u_2(x)$ such that:

$$\begin{cases} u_1(x) = u(x); \\ u_2(x) = \frac{du(x)}{dx}. \end{cases} \tag{13}$$

This position was sufficient to carry out the above mentioned transformation.

Remark 4.1. Equalities (13), besides having a mathematical meaning, are worth investigating from the physical point of view. $u_1(x)$ and $u_2(x)$ represent, $\forall x \in [-L, L]$, the profile of the membrane and its slope, respectively. In other words, the equalities (13) allow us to rewrite (4) into a system of equations where the unknown functions are the position of the membrane and the rate of change of the position of the membrane along the device, respectively. This allows us to open an important scenario for the study of equilibrium positions and to investigate their stability.

Taking into account the position (13), after simple mathematical steps, model (4) became:

$$\begin{cases} \frac{du_1(x)}{dx} = u_2(x); \\ \frac{du_2(x)}{dx} = -\frac{1}{\theta\lambda^2}(1 + (u_2(x))^2)^3(1 - u_1(x) - d^*)^2 \\ u_1(-L) = u_1(L) = 0 \end{cases} \tag{14}$$

in $[-L, L]$ which, as above specified, represents a system of first-order differential equations in normal form with $u_1(x)$ and $u_2(x)$ as unknown functions.

4.2. *Equilibrium Configurations*

As already specified, to obtain the equilibrium configuration for (14) (and therefore for model (4)), it is enough to impose $\frac{du_1(x)}{dx} = 0$ and $\frac{du_2(x)}{dx} = 0$ [31]. Then, we can write:

$$\begin{cases} \frac{du_1(x)}{dx} = u_2(x) = 0; \\ \frac{du_2(x)}{dx} = -\frac{1}{\theta\lambda^2}(1 + (u_2(x))^2)^3(1 - u_1(x) - d^*)^2 = 0 \end{cases} \tag{15}$$

in $[-L, L]$, with $\theta\lambda^2 \neq 0$. We achieve the following equilibrium point $(u_1^0, u_2^0) = (1 - d^*, 0)$. Then, we have tested if the only equilibrium point obtained, (u_1^0, u_2^0), deemed critical, is also stable. To obtain this information, since system (14) is non-linear, we applied the consolidated linearization technique of a first-order differential system around the equilibrium position (technique know as the first Lyapunov criterion).

Remark 4.2. The first Lyapunov criterion is based on the linearization of the non-linear system (14) in the neighborhood of the equilibrium state whose stability is to be studied. A theory of analysis system typical of linear systems can be applied to the linear system thus obtained. The information obtained in this way allows to draw conclusions on the behavior of the starting non-linear system in a neighborhood of the considered equilibrium state.

5. On the Stability of the Equilibrium Configuration

5.1. *Linearization of the Non-Linear First-Order Differential Equations System Around the Equilibrium Configuration*

We first note that the system (14) can be written in the general form (12) where, in our case,

$$\overline{f}(u_1(x), u_2(x)) = u_2(x); \quad \overline{g}(u_1(x), u_2(x)) = -\frac{1}{\theta\lambda^2}(1 + (u_2(x))^2)^3(1 - u_1(x) - d^*)^2. \tag{16}$$

Remark 5.1. The general model (12) can be written in matrix notation. With this goal, placing

$$\mathbf{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad \dot{\mathbf{u}}(x) = \begin{pmatrix} \frac{du_1(x)}{dx} \\ \frac{du_2(x)}{dx} \end{pmatrix} \quad \mathbf{f}(x, u(x)) = \begin{pmatrix} \overline{f}(u_1(x), u_2(x)) \\ \overline{g}(u_1(x), u_2(x)) \end{pmatrix}, \tag{17}$$

system (12) can be written as follows:

$$\dot{\mathbf{u}}(x) = \mathbf{f}(x, u(x)), \tag{18}$$

which represents the algebraic formulation of the problem (4) in normal form.

5.1.1. *A Suitable Change of Variables*

In order to linearize the system, we used the following change of variable:

$$u_1(x) = u_1^0 + \epsilon\xi(x); \quad u_2(x) = u_2^0 + \epsilon\eta(x) \tag{19}$$

with a small enough ϵ.

5.1.2. *Taylor Series Development up to the First-Order*

From system (12), taking into account (19) and considering that u_1^0 e u_2^0 do not depend on x, we can write:

$$\begin{cases} \frac{du_1(x)}{dx} = \epsilon\frac{d\xi(x)}{dx} = \overline{f}(u_1(x), u_2(x)) \\ \frac{du_2(x)}{dx} = \epsilon\frac{d\eta(x)}{dx} = \overline{g}(u_1(x), u_2(x)). \end{cases} \tag{20}$$

Then developing in Taylor series $\overline{f}(u_1(x), u_2(x))$, $\overline{g}(u_1(x), u_2(x))$ and neglecting the terms of order higher than the first, we can write:

$$\begin{cases} \epsilon\frac{d\xi(x)}{dx} = \overline{f}(u_1^0 + \epsilon\xi(x), u_2^0 + \epsilon\eta(x)) \approx \overline{f}(u_1^0, u_2^0) + \epsilon\frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_1}\xi(x) + \epsilon\frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_2}\eta(x) + o(\tau) \\ \epsilon\frac{d\eta(x)}{dx} = \overline{g}(u_1^0 + \epsilon\xi(x), u_2^0 + \epsilon\eta(x)) \approx \overline{g}(u_1^0, u_2^0) + \epsilon\frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_1}\xi(x) + \epsilon\frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_2}\eta(x) + o(\tau) \end{cases} \tag{21}$$

where $\tau = \epsilon\sqrt{\xi^2 + \eta^2}$. Taking into account that $\overline{f}(u_1^0, u_2^0) = \overline{g}(u_1^0, u_2^0) = 0$, and neglecting the infinitesimal terms of order higher than τ, we achieve:

$$\begin{cases} \frac{d\xi(x)}{dx} = \frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_1}\xi(x) + \frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_2}\eta(x) \\ \frac{d\eta(x)}{dx} = \frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_1}\xi(x) + \frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_2}\eta(x) \end{cases}, \tag{22}$$

which represents a system of first-order ordinary differential equations.

Remark 5.2. Even system (22), as for system (12), admits a matrix representation. In particular, by placing:

$$\mathbf{z} = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}, \quad \dot{\mathbf{z}} = \begin{pmatrix} \frac{d\xi(x)}{dx} \\ \frac{d\eta(x)}{dx} \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_1} & \frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_2} \\ \frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_1} & \frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_2} \end{pmatrix} \tag{23}$$

the system (22) takes the following matrix form:

$$\dot{\mathbf{z}} = A\mathbf{z}. \tag{24}$$

Finally, it seems useful to underline the fact that system (24) is the linearized form of the non-linear system around the equilibrium position (18).

In our case, taking into account (16), the elements of A are detailed as follows:

$$\begin{cases} \frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_1} = 0; & \frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_2} = 1 \\ \frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_1} = \frac{2}{\theta\lambda^2}(1 - d^* - 1 + d^*) = 0; & \frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_2} = -\frac{6u_2}{\theta\lambda^2}(1 - d^* - 1 + d^*)^2 = 0. \end{cases} \tag{25}$$

Thus, system (22) (that is, also the system (24)) becomes:

$$\begin{cases} \frac{d\xi(x)}{dx} = \eta \\ \frac{d\eta(x)}{dx} = 0 \end{cases}, \tag{26}$$

which represents the system (14) linearized. System (26), solved, provides $\xi(x) = k_1 + kx$ nd $\eta(x) = k$ with k_1 and k integration constant.

Remark 5.3. Equations (21) make sense because u_1 e u_2, as proved in [5], are analytical functions allowing the linearization through the writing of $\frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_1}$, $\frac{\partial\overline{f}(u_1^0,u_2^0)}{\partial u_2}$, $\frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_1}$ and $\frac{\partial\overline{g}(u_1^0,u_2^0)}{\partial u_2}$.

5.2. *Study of the Stability of the Equilibrium Configuration*

5.2.1. *Calculation of the Eigenvalues of the Coefficient Matrix and Computation of the Algebraic Multiplicity*

Taking into account (25), $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and its eigenvalues are $\zeta_1 = \zeta_2 = 0$. Therefore, A has a single non-simple eigenvalue because its algebraic multiplicity is equal to 2.

5.2.2. *Orbits*

From the theory of linear systems of differential equations, the accepted solutions are of the form [31]:

$$\begin{cases} \xi(x) = axe^{\lambda x} \\ \eta(x) = bxe^{\lambda x}. \end{cases} \tag{27}$$

Form the system (27), we obtain $\xi(x) = \frac{a}{b}\eta(x)$ which represents, with a and b variable, a proper bundle of straight lines with center $(0, 0)$ (see Fig. 1b). The physical-mathematical meaning of the orbits is of interest. Indicating by $\mathbf{u}_{eq} = (u_1^0, u_2^0)$ the equilibrium position, it is stable according to Lyapunov's theory if, for each neighborhood U of $\mathbf{u}_{eq}$, there exists a neighborhood $B \subset U$ such that the orbits starting from the internal point of B remain inside U. The five following definitions are explicit the above concept.

Definition 5.1 (Stable Equilibrium Point). *$\forall \epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, if $\|\mathbf{u}(0) - \mathbf{u}_{eq}\| < \delta$, then $\forall x \geq 0$, it follows that $\|\mathbf{u}(x) - \mathbf{u}_{eq}\| < \epsilon$.*

Definition 5.2 (Attractive Equilibrium Point). *$\mathbf{u}_{eq}$ is said to be attractive if there exists a neighborhood U of $\mathbf{u}_{eq}$ such that for each orbit $\mathbf{u}(x)$ starting from an internal point of U, the limit $\lim_{t\to+\infty} \mathbf{u}(x) = \mathbf{u}_{eq}$ yields*

Definition 5.3 (Asymptotically Stable Equilibrium Point). *$\mathbf{u}_{eq}$ is said to be asymptotically stable if it is stable and attractive as well. In other words, there exists $\delta > 0$ such that if $\|\mathbf{u}(0) - \mathbf{u}_{eq}\| < \delta$ then $\lim_{t\to+\infty} \|\mathbf{u}(x) - \mathbf{u}_{eq}\| = 0$.*

Definition 5.4 (Exponentially Stable Equilibrium Point). *$\mathbf{u}_{eq}$ is said to be an exponentially stable equilibrium point if it is asymptotically stable and, moreover, there exists α, β, $\delta > 0$ such that, if $\|\mathbf{u}(0) - \mathbf{u}_{eq}\| < \delta$, we have $\|\mathbf{u}(x) - \mathbf{u}_{eq} \leq \alpha\|\mathbf{u}(0) - \mathbf{u}_{eq}\|e^{-\beta x}$.*

Definition 5.5 (Unstable Equilibrium Point). *An equilibrium point is said to be unstable if there exists a neighborhood U of $\mathbf{u}_{eq}$ such that, however one chooses a neighborhood B of $\mathbf{x}_{eq}$ contained in U, one can always find an initial position $\mathbf{x} \in B$ whose orbit moves away from $\mathbf{x}_{eq}$ enough to exit from U.*

Examining Fig. 1b it is clear that, by Definition 5.5, $\mathbf{u}_{eq} = (u_1^0, u_2^0)$ is an unstable equilibrium point because there does not exist a neighborhood U of $\mathbf{u}_{eq}$ such that, however $B \in U$ is chosen, the orbits are not contained in U.

5.2.3. *Evaluation of the stability of the only point of equilibrium*

To study the stability of the linearized system around the equilibrium position we use the following result known in the literature.

Lemma 5.1. *In a system of first-order linear differential equations writable in the form (24), if there is at least one eigenvalue with a positive real part or a not simple eigenvalue with zero real part, then the system is unstable. Graphically, this occurrence means that the orbits diverge from the origin (see Fig. 1b) [31].*

Proposition 5.1. *System (26) is not stable.*

Proof. In this case, and with simple calculations, we obtain $\zeta_1 = \zeta_2 = \zeta = 0$. Thus, we deduce that the only eigenvalue is not simple as its algebraic multiplicity is equal to 2. Then, by Lemma 5.1, the system (26) is not stable. □

The following result holds.

Lemma 5.2. *If an equilibrium point of a linearized system $\dot{\mathbf{z}} = A\mathbf{z}$ is unstable, then the equilibrium point (u_1^0, u_2^0) of the starting non-linear system $\dot{\mathbf{u}}(x) = \mathbf{f}(x)$ is also unstable [31].*

Proposition 5.2. *Let us consider system (14) whose only equilibrium position turns out to be $(u_1^0, u_2^0) = (1 - d^*, 0)$. Then, this solution is unstable.*

Proof. It follows from Lemma 5.2. □

Remark 5.4. From the present study, we determined that the membrane, under the effect of V applied externally, deforms reaching a distance equal in $u(0)$ equal to $1 - d^*$. However, this configuration of the membrane is of unstable equilibrium.

Remark 5.5. The transition between stability and instability of an equilibrium in a membrane MEMS device can have multiple causes. For example, instability phenomena can occur when the non-equilibrium state is related to the restriction failure mode occurring in the operation of MEMS devices. Electrostatically, the instability of the membrane can be affected by small disturbances of the p_{el} distribution in the device influencing: (a) the equilibrium path; (b) the number and location of unstable positions; and (c) the post-instability behavior. In addition, p_{el} depends on the V applied. Specifically, [24]:

$$p_{el} = \frac{f_{el}}{A} = \underbrace{\frac{1}{2}\frac{\epsilon_0 A V^2}{(d-u(x))^2}}_{\text{electrostatic force}} \frac{1}{A} = \frac{1}{2}\frac{\epsilon_0 V^2}{(d-u(x))^2}, \tag{28}$$

in which f_{el} is the electrostatic force inside the device and A is the measure of the curve representing the membrane. Moreover, snapping behavior can occur if the system is repeatedly charged and discharged electrically [32]. Finally, marked phenomena of instability can occur due to the Casimir effect regardless of the V applied. [**?**].

By Remark 5.4, one needs to know the $\sup\{V\}$ to ensure that the membrane does not go beyond the deflection equal to $1-d^*$, ensuring the achievement, at most, of the equilibrium position. On the other hand, the membrane has a certain inertia in moving under the action of external V, so it is also worth asking which will be the minimum value of V to allow the deflection of the membrane. Moreover, as highlighted in Remark 5.5, the knowledge of the range of possible value of V for a device allows us to understand, by (28), whether the p_{el} achieved leads to instability phenomena.

Finally, to know the range of possible values of V has repercussions for knowing the range of possible values of λ^2 (proportional to the square of V, as specified in (8)), which serve, on one hand, as a tuning parameter for the device [11] and, on the other hand, to ensure the convergence of the numerical procedure to recover the profile of the membrane [7,8].

6. On the Admissible Values of the Applied Voltage

Before analyzing the problem, let us introduce the definition of the pull-in voltage λ^* [11].

Definition 6.1. Let us consider the problem (27). The pull-in voltage, λ^*, is defined as [11]:

$$\lambda^* = \sup\left\{\lambda^2 \quad \ni' \quad \text{problem (1) admits at least one solution}\right\}. \tag{29}$$

In our case, as, we are studying the problem (4), we define a special pull-in voltage as follows.

Definition 6.2. We define the special pull-in voltage, $(\lambda^*)_{\text{special}}$, the quantity:

$$(\lambda^*)_{\text{special}} = \sup\left\{\lambda^2 \quad \ni' \quad \text{problem (4) admits at least one solution}\right\}. \tag{30}$$

Remark 6.1. In (8), in the case of a 1D electrostatic membrane MEMS, T can be considered as a constant [11]. Thus, with the same V, the stiffer the membrane is, the greater T will be, so that λ^2 decreases.

As λ^2 is involved in the definition of the pull-in voltage, we wonder what the maximum value of V will be for which the problem (4) admits at least one solution. In other words, we ask what will be the maximum value of V that is able to move the membrane up to the safety distance. Clearly, considering the $\sup\{V\}$, we automatically have to consider $\sup\{\lambda^2\}$. Furthermore, the direct proportionality between V^2 and λ^2 suggests that there is a value of V linked to $(\lambda^*)_{\text{special}}$.

Proposition 6.1. *Let us consider the problem (4). Let (8) be the link between $(\lambda^*)_{special}$ and V^2. Then, with $C = \frac{\epsilon_0 L^2}{2d^3 T}$, the following condition holds:*

$$(\lambda^*)_{special} < C \sup\{V^2\}, \tag{31}$$

Proof. By $\lambda^2 = \frac{\epsilon_0 L^2 V^2}{d^3 T}$ [11], and considering the (30), the (31) follows. □

Remark 6.2. We immediately notice from Proposition 6.1 that, in our case, $\sup\{V^2\}$ is more salient than $(\lambda^*)_{\text{special}}$. In fact, V is a global parameter which, on the one hand, is directly measurable by means of a voltmeter and, on the other, is uniquely selectable by means of an electric voltage generator.

6.1. *Condition of existence of at least one solution for the problem* (4)

To derive the range of admissible values of V we must first ensure the condition of existence of at least one solution for the problem (4). In [5,7,8] this condition of existence was achieved. In particular, indicating with $\overline{\lambda^2}$ the $\min\{\lambda^2\}$ to overcome the membrane mechanical inertia, this condition assumes the following known form:

$$1 + H^6 < \frac{H\theta\overline{\lambda}^2}{4(1-d^*)L} \tag{32}$$

where $H = \sup_{[-L,L]}\left\{\left|\frac{du(x)}{dx}\right|\right\}$.

Remark 6.3. In [5], by means of a numerical technique based on the Newton–Raphson procedure, to guarantee the existence of the solution of the problem (4), it is sufficient that $H = \sup_{[-L,L]}\left\{\left|\frac{du(x)}{dx}\right|\right\} = 99$ corresponding, in the dimensionless condition, to 88.92°.

6.2. *The Minimum Values of V to Overcome the Mechanical Inertia of the Membrane*

Proposition 6.2. *Let us consider the 1D membrane MEMS device whose analytical model is represented by the system of first-order differential equations* (14)*. If the condition of existence and uniqueness* (32) *for its solution holds, then the minimum V needed to overcome the membrane mechanical inertia holds*

$$(V_{min})_{inertia} > \sqrt{\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}}\sqrt{\frac{1+H^6}{H}}. \tag{33}$$

Proof. In fact, taking into account that $\overline{\lambda^2} < \lambda^2$, from (32) it follows that:

$$\lambda^2 > \frac{4L(1-d^*)(1+H^6)}{H\theta}. \tag{34}$$

Substituting (34) in (8), we achieve:

$$V > \frac{\sqrt{2Td^3(1-d^*)(1+H^6)}}{\sqrt{\epsilon_0 LH\theta}} = \sqrt{\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}}\sqrt{\frac{1+H^6}{H}}. \tag{35}$$

$\sqrt{\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}}\sqrt{\frac{1+H^6}{H}}$ in inequality (35) is less than the V necessary to achieve the mechanical inertia of the membrane. Then, it makes sense to write:

$$(V_{min})_{inertia} > \sqrt{\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}}\sqrt{\frac{1+H^6}{H}}. \tag{36}$$

□

6.3. *The Maximum allowed V value so that the Membrane does not Touch the Upper Plate*

We propose the following three propositions.

Proposition 6.3. *For the MEMS device, indicating with u_0 the deflection in the center of the membrane, the following inequality holds:*

$$u_0 \leq \frac{k\epsilon_0 V^2}{2d^*}. \tag{37}$$

Proof. *The link between V and $u(x)$ can be specified as follows [11,24]:*

$$u(x) \leq \frac{k\epsilon_0 V^2}{2d^*}\left\{1 - \left(\frac{x}{L}\right)^2\right\}. \tag{38}$$

Taking into account that $\frac{1}{d-u(x)} < \frac{1}{d^*}$, we obtain the (37) □

Proposition 6.4. *For the MEMS device under study, $\sqrt{k}$ assumes the following expression:*

$$\sqrt{k} = \frac{\sqrt{p}}{2\sqrt{p_{el}}}. \tag{39}$$

Proof. $u(x)$ can be written as follows [11] $u(x) = -\frac{p}{4}(x^2 - 1)$ from which $u_0 = u(0) = \frac{p}{4}$, so that

$$\sqrt{u_0} = \frac{\sqrt{p}}{2}. \tag{40}$$

As explicated in [24], $u_0 = kp_{el}$, thus

$$\sqrt{k} = \frac{\sqrt{u_0}}{\sqrt{p_{el}}}. \tag{41}$$

Combining (40) and (41), we find equality (39). □

Proposition 6.5. *For the MEMS device under study, the following inequality holds:*

$$\frac{k\epsilon_0 V^2}{2d^*} < 1 - d^*. \tag{42}$$

Proof. It is known that (see [11]) $k = \frac{p}{4p_{el}}$ and p_{el} is computed by (28). Then, taking into account both (37) and (28), we can write:

$$u_0 \leq \frac{k\epsilon_0 V^2}{2d^*} = \frac{p}{8p_{el}}\frac{\epsilon_0 V^2}{d^*} = \frac{p\epsilon_0 V^2}{8d^*}\frac{2(d-u(x))^2}{\epsilon_0 V^2} = \frac{p}{4d^*}(d-u(x))^2 \tag{43}$$

However, from a mechanical point of view, we can write [11] $u(x) = -\frac{p}{4}(x^2 - 1)$ from which $p = \frac{4u(x)}{(1-x^2)}$. Then:

$$u_0 \leq \frac{u(x)(d-u(x))^2}{(1-x^2)d^*}. \tag{44}$$

We easily observe that $\frac{1}{1-x^2} = \frac{1}{2}\left\{\frac{1}{1-x} + \frac{1}{1+x}\right\}$ so that inequality (44) becomes:

$$u_0 \leq \frac{u(x)(d-u(x))^2}{(1-x^2)d^*} = \frac{u(x)(d-u(x))^2}{2d^*}\left\{\frac{1}{1-x} + \frac{1}{1+x}\right\}. \tag{45}$$

Moreover, $-L \leq x \leq L$ [24], with $L \ll 1$, so that $1 - L \leq 1 + x \leq 1 + L$. Then

$$\frac{1}{1+x} \leq \frac{1}{1-L}. \tag{46}$$

Analogously, $1 - L \leq 1 - x \leq 1 + L$ from which

$$\frac{1}{1-x} \leq \frac{1}{1-L}. \tag{47}$$

Finally, taking into account both inequalities (46) and (47), inequality (45) becomes:

$$u_0 \leq \frac{u(x)(d-u(x))^2}{(1-x^2)d^*} = \frac{u(x)(d-u(x))^2}{2d^*}\left\{\frac{1}{1-x} + \frac{1}{1+x}\right\} \leq \frac{u(x)(d-u(x))^2}{d^*(1-L)}. \tag{48}$$

Finally, taking into account both (43) and (48), let us consider the following chain of inequalities:

$$u_0 \leq \frac{k\epsilon_0 V^2}{2d^*} \leq \frac{u(x)}{d^*(1-L)}(d-u(x))^2. \tag{49}$$

If we prove that

$$\frac{u(x)}{d^*(1-L)}(d-u(x))^2 < 1 - d^*, \tag{50}$$

then we prove inequality (42), because it would occur, considering both (49) and (50), that

$$u_0 \leq \frac{k\epsilon_0 V^2}{2d^*} \leq \frac{u(x)}{d^*(1-L)}(d-u(x))^2 < 1 - d^*. \tag{51}$$

Let us suppose, absurdly, that

$$\frac{u(x)}{d^*(1-L)}(d-u(x))^2 \geq 1 - d^* \tag{52}$$

from which we can write:

$$u(x)(d-u(x))^2 \geq d^*(1-L)(1-d^*). \tag{53}$$

Three cases can occur.

Case 1. If the membrane is at rest, $u(x) = 0$ and, from (53), it follows that $0 \geq d^*(1-L)(1-d^*)$ which represents an incorrect condition because $d^*(1-L)(1-d^*)$ is a positive quantity. Then, in this case, inequality (50) is verified, so that also the (42) is verified.

Case 2. If we consider the maximum allowed deformation, $u(x) = d - d^*$, from (53), we can write:

$$d - d^* \geq \frac{(1-L)(1-d^*)}{d^*}. \tag{54}$$

However, from a constructive point of view, typically $d^* = \frac{d}{10}$, thus, (54) becomes

$$\frac{9}{10}d \geq \frac{10(1-L)\left(1-\frac{d}{10}\right)}{d} \tag{55}$$

from which

$$9d^2 + 10d(1-L) - 100(1-L) \geq 0 \tag{56}$$

so that

$$d \geq \frac{-5(1-L)+5\sqrt{(1-L)(37-L)}}{9} \tag{57}$$

which represents a physically false condition ($d \approx 10^{-9}$). Then, also in this case, inequality (52) is false so that inequality (50) is true and (42) is verified.

Case 3. For any other $u(x)$ value, (50) is also verified. In fact, if, absurdly, the inequality (52) holds, and taking into account that $d^* = \frac{d}{10}$, we can write:

$$\frac{10u(x)}{d(1-L)}(d-u(x))^2 > 1 - \frac{d}{10}. \tag{58}$$

However, we easily observe that $d > u(x)$ from which $10 > \frac{10u(x)}{d}$ and again, accounting for (58), it follows that

$$u(x) < d - \sqrt{\frac{(1-L)(10-d)}{100}}. \tag{59}$$

If, absurdly,

$$d > \sqrt{\frac{(1-L)(10-d)}{100}} \tag{60}$$

it is satisfied if, with the usual values for L and d and not considering the negative root as it has no physical meaning, $d \gg 10^{-9}$. As this range of d, from the constructive point of view, is not valid, it follows that the inequality (60) is absurd, so that it follows:

$$d \leq \sqrt{\frac{(1-L)(10-d)}{100}}. \tag{61}$$

Then, in (59), $u(x)$ results in a lower than negative quantity (physically impossible). From which we deduce that inequality (42) is, also in this case, verified. Then, also the (42) is verified. This result concludes the proof. □

Propositions 6.3, 6.4, and 6.5 have been presented and proven, and we are able to easily evaluate the $(V_{max})_{permissible}$.

Proposition 6.6. *Let the membrane MEMS device whose analytical model is represented by the system of differential equation of the first-order (14). Then, the maximum permissible $(V_{max})_{permissible}$ is increased as follows so that the membrane does not touch the upper plate:*

$$(V_{max})_{permissible} < \sqrt{\frac{2(1-d^*)d^*}{k\epsilon_0}}. \tag{62}$$

Proof. Trivially follows from (42). □

We propose the following.

Proposition 6.7. *For the MEMS device under study, the following expression yields:*

$$\sqrt{L\theta} = \frac{L}{d}\sqrt{\frac{\epsilon_0 L}{2dT}}. \tag{63}$$

Proof. Considering both (8) and (10), we obtain:

$$\theta = \frac{\theta\lambda^2}{\lambda^2} = \frac{\epsilon_0^2 L^4}{d^6 T^2} \frac{V^4}{(1-u(x))^2 |\mathbf{E}|^2} \frac{2d^3 T}{\epsilon_0 V^2 4L^2} = \frac{\epsilon_0 L^2 V^2}{2d^3 T |\mathbf{E}|^2 (1-u(x))^2}. \tag{64}$$

However, electrostatically

$$|\mathbf{E}|^2 = \frac{V^2}{(1-u(x))^2}. \tag{65}$$

Then, substituting (65) into (64), we achieve the (63). □

Then, we are ready to present the following result.

Proposition 6.8. *For the MEMS device under study, the following inequality holds:*

$$\sqrt{\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}} \sqrt{\frac{1+H^6}{H}} < \sqrt{\frac{2(1-d^*)d^*}{k\epsilon_0}}. \tag{66}$$

Proof. Let us suppose, absurdly, that

$$\sqrt{\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}} \sqrt{\frac{1+H^6}{H}} > \sqrt{\frac{2(1-d^*)d^*}{k\epsilon_0}}. \tag{67}$$

from which, taking into account equality (63), we obtain:

$$\frac{2d^3 T}{L\sqrt{\epsilon_0 L}} \sqrt{\frac{1+H^6}{H}} > \frac{\sqrt{d^*}}{\sqrt{k}}. \tag{68}$$

From (39), observing that $u_0 \leq d - d^*$ and therefore $\frac{1}{\sqrt{u_0}} \geq \frac{1}{\sqrt{d-d^*}}$, we can write:

$$\frac{1}{\sqrt{k}} \geq \frac{\sqrt{p_{el}}}{\sqrt{d-d^*}}. \tag{69}$$

Then, inequality (68), taking into account (69) and considering that $d^* = \frac{d}{10}$, can be written as follows:

$$\frac{\sqrt{p_{el}}}{T} \leq \frac{6d^3}{L\sqrt{\epsilon_0 L}} \sqrt{\frac{1+H^6}{H}}. \tag{70}$$

Considering that $H \approx 99$ [5], it follows that $\sqrt{\frac{1+H^6}{H}} \approx 97 \cdot 10^4$. As $L = 10^{-6}$ (the device is a micro-device, and therefore its length is $\approx 10^{-6}$) and $\epsilon_0 = 8.85 \cdot 10^{-12}$, while, as known, $d \approx 10^{-9}$ (usually the height of the device is 10^{-3} of the width). Thus, the inequality (70) assumes the form $\frac{\sqrt{p_{el}}}{T} \leq 0.95 \cdot 10^{-6}$. In other words, the mechanical tension T of the membrane would be a very large value as if the membrane were extremely rigid. This condition is physically not compatible with the usual membranes used in MEMS devices. Then, inequality (67) is false; therefore, inequality (66) yields. □

Proposition 6.9. *Consider the membrane MEMS device whose analytical model is represented by the system of first-order differential equations* (14)*. Then, the range of admissible values for V so that, on the one hand, the mechanical inertia of the membrane overcomes and, on the other, the membrane does not touch the upper plate, is as follows:*

$$\sqrt{\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}} \sqrt{\frac{1+H^6}{H}} < V < \sqrt{\frac{2(1-d^*)d^*}{k\epsilon_0}}. \tag{71}$$

Proof. (71) immediately follows from Propositions 6.2 to 6.6. □

6.4. *The Relationship between* $(V_{min})_{inertia}$ *and* $(V_{max})_{permissible}$

Proposition 6.10. *The relationship between* $(V_{min})_{inertia}$ *and* $(V_{max})_{permissible}$ *is the following:*

$$(V_{min})_{inertia} > (V_{max})_{permissible} \frac{k_2 d\sqrt{d}}{2L\sqrt{\epsilon_0 L}} \sqrt{\frac{1+H^6}{H}} T. \tag{72}$$

Proof. From (62) we can write:

$$(V_{max})_{permissible} < \sqrt{\frac{2(1-d^*)}{\epsilon_0}} \sqrt{\frac{d^*}{k}} = \sqrt{\frac{2(1-d^*)}{\epsilon_0}} \sqrt{\frac{d}{10k}} \tag{73}$$

from which:

$$\sqrt{\frac{2(1-d^*)}{\epsilon_0}} > (V_{max})_{permissible} \sqrt{\frac{10k}{d}}. \tag{74}$$

In addition, from (33), and taking into account (73), it makes sense to write:

$$(V_{min})_{inertia} > (V_{max})_{permissible} \frac{d\sqrt{10k}}{\sqrt{L\theta}} \sqrt{\frac{1+H^6}{H}} \sqrt{T}. \tag{75}$$

Taking into account Proposition 6.4, from [24], $p = k_1 p_{el}$, with $k_1 = 4k$ and $k_2 = 2\sqrt{k}$, (75) becomes:

$$(V_{min})_{inertia} > (V_{max})_{permissible} \frac{k_2}{2} \frac{d\sqrt{10}}{\sqrt{L\theta}} \sqrt{\frac{1+H^6}{H}} \sqrt{T}. \tag{76}$$

In addition, taking into account Proposition 6.7, (76) can be written as follows:

$$(V_{min})_{inertia} > (V_{max})_{permissible} \frac{k_2 d^2 \sqrt{10}\sqrt{d}}{\sqrt{2}L\sqrt{\epsilon_0 L}} \sqrt{\frac{1+H^6}{H}} T. \tag{77}$$

□

Remark 6.4. Considering that $d = 10^{-9}$, $\epsilon_0 = 8.85 \cdot 10^{-12}$ and $L = 10^{-6}$, then $\frac{d^2\sqrt{10}\sqrt{d}}{\sqrt{2}L\sqrt{\epsilon_0 L}} \sqrt{\frac{1+H^6}{H}} \approx 8.15 \cdot 10^{-16} \ll 1$. Then, inequality (77) becomes

$$(V_{min})_{inertia} > (V_{max})_{permissible} \cdot 0.023 k_2 T. \tag{78}$$

As known [15], $k_2 \approx 1$ because, physically, $p_{el} \simeq p$ and, in addition, T is of the order of 1000 Pa. Then inequality (78) can be written as follows:

$$(V_{min})_{inertia} > (V_{max})_{permissible} \cdot 0.023 k_2 T \simeq (V_{max})_{permissible} \cdot 0.023. \tag{79}$$

so that inequality (77) makes physical sense.

Remark 6.5. Inequality (77) provides a useful physical interpretation. In it, $\frac{d^2\sqrt{10}\sqrt{d}}{4L\sqrt{\epsilon_0 L}} \sqrt{\frac{1+H^6}{H}}$ can be considered a constant only depending on the geometrical parameters of the device. In other words, we can write $G = G(L, d) = \frac{d^2\sqrt{10}\sqrt{d}}{\sqrt{2}L\sqrt{\epsilon_0 L}} \sqrt{\frac{1+H^6}{H}}$ so that inequality (77) becomes:

$$(V_{min})_{inertia} > (V_{max})_{permissible} G k_2 T \tag{80}$$

from which, considering that $k_2 \approx 1$, it follows that:

$$\frac{(V_{min})_{inertia}}{(V_{max})_{permissible}} > GT. \tag{81}$$

Thus, having chosen the material constituting the membrane (in other words, having fixed T), $\frac{(V_{min})_{inertia}}{(V_{max})_{permissible}}$ depends on a quantity (i.e., G) that changes with the geometry of the device. If, on the other hand, we choose the geometry of the device (in other words, we fix the constant G), it is the material that constitutes the membrane that determines $\frac{(V_{min})_{inertia}}{(V_{max})_{permissible}}$. Fig. 2a displays $\frac{(V_{min})_{inertia}}{(V_{max})_{permissible}}$ versus T according to (81).

7. Some Remarks about the Potential Energy in the Device

When the membrane is at rest, the distance between itself and the upper plate is equal to d, so that the electrostatic capacity of the device is $C = \epsilon_0 \frac{2L}{d}$. Then, the potential energy of the device, when the membrane is at rest, is $W_{initial} = \frac{1}{2}CV^2 = \frac{\epsilon_0 L}{d}V^2$. When the membrane deforms, the electrostatic capacity becomes $C = \epsilon_0 \int_{-L}^{+L} \frac{dx}{d-u(x)}$. Then, in this case, the final potential energy can be computed as $W_{final} = \frac{1}{2}CV^2 = \frac{1}{2}\epsilon_0 V^2 \int_{-L}^{+L} \frac{dx}{d-u(x)}$. Therefore, the total variation of the potential energy, ΔW, is:

(82) $$\Delta W = W_{final} - W_{initial} = \epsilon_0 V^2 \Big\{ \frac{1}{2} \int_{-L}^{+L} \frac{dx}{d-u(x)} - \frac{L}{d} \Big\}.$$

We preliminary observe that $d - u(x) \geq d - d^*$ from which $\frac{1}{d-u(x)} \leq \frac{1}{d-d^*}$. Therefore, from (82), we can write:

(83) $$\Delta W = \epsilon_0 V^2 \Big\{ \frac{1}{2} \int_{-L}^{+L} \frac{dx}{d-u(x)} - \frac{L}{d} \Big\} \leq \epsilon_0 V^2 \Big\{ \frac{L}{d-d^*} - \frac{L}{d} \Big\} = \epsilon_0 V^2 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\}.$$

But from (71) we can write $\frac{2Td^3(1-d^*)}{\epsilon_0 L\theta}\frac{1+H^6}{H} < V^2 < \frac{2(1-d^*)d^*}{k\epsilon_0}$ so that, taking into account (83), it follows:

(84) $$\Delta W \leq \epsilon_0 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\} V^2 < 2\epsilon_0 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\} \frac{(1-d^*)d^*}{k} = \frac{2(d^*)^2(1-d^*)L}{kd(d-d^*)}.$$

Moreover, from (36), $V^2 > (V_{min})^2_{inertia} > \frac{8Td^3(1-d^*)}{\epsilon_0 L\theta}\frac{1+H^6}{H}$ from which we obtain:

(85) $$\Delta W > \epsilon_0 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\} V^2 > \frac{8d^*Td^3(1-d^*)}{d(d-d^*)\theta}\frac{1+H^6}{H}$$

Finally, combining (84) and (85), we achieve the range of the admissible values for ΔW:

(86) $$\frac{8d^*Td^3(1-d^*)}{d(d-d^*)\theta}\frac{1+H^6}{H} < \Delta W < \frac{2(d^*)^2(1-d^*)L}{kd(d-d^*)}$$

Remark 7.1. $\frac{8d^*Td^3(1-d^*)}{d(d-d^*)\theta}\frac{1+H^6}{H}$ in (86) derives from the $(V_{min})_{inertia}$ and depends on both T and θ, i.e. on the electromechanical properties of the membrane (θ) and on the mechanical tension of the membrane before the deformation (T). In addition, $\frac{2(d^*)^2(1-d^*)L}{kd(d-d^*)}$ derives from $(V_{max})_{permissible}$ and depends on the ration between the mechanical pressure and electrostatic pressure (k).

The (86), considering that in dimensionless conditions $d^* = \frac{d}{10}$, $\frac{1+H^6}{H} \approx 97 \cdot 10^4$, $d = 1$ and $L = 0.5$, becomes:

(87) $$\frac{7524T}{\theta} < \Delta W < \frac{0.005}{k}.$$

Figure 2b displays the area of possible values for ΔW. Obviously, once T is fixed (i.e. the method of anchoring the membrane on the lower plate is fixed), as the θ increases, the blue straight line rotates clockwise, while the horizontal line decreases as k increases.

7.1. *On the Value of V which maximizes ΔW*

Proposition 7.1. $V = \sqrt{\frac{2dd^*(\epsilon_0+\sqrt{\epsilon_0})}{k\epsilon_0^2}}$ *maximizes ΔW. In addition:*

(88) $$V = \sqrt{\frac{2dd^*(\epsilon_0+\sqrt{\epsilon_0})}{k\epsilon_0^2}} < \sup\{(V_{max})_{permissible}\} = \sqrt{\frac{2(1-d^*)d^*}{k\epsilon_0}}$$

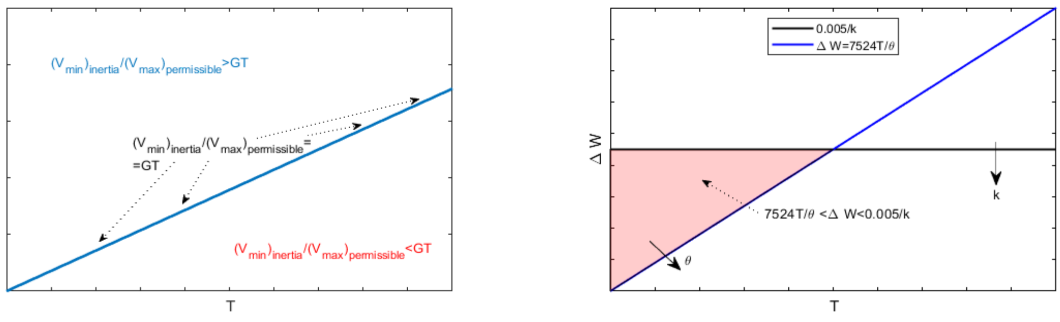


Figure 2. (a) $\frac{(V_{min})_{inertia}}{(V_{max})_{permissible}}$ versus T. (b) Area of possible values for ΔW according to the (87).

Proof. Taking into account the inequality (38), we can write $d-u(x) \geq d-\frac{k\epsilon_0 V^2}{2d^*}\left(1-\left(\frac{x}{L}\right)^2\right) > \frac{2dd^*-k\epsilon_0 V^2}{2d^*}$ from which $\frac{1}{d-u(x)} < \frac{2d^*}{2dd^*-k\epsilon_0 V^2}$ so that

$$C = \epsilon_0 \int_{-L}^{+L} \frac{dx}{d-u(x)} < \epsilon_0 \int_{-L}^{+L} \frac{2d^*}{2dd^*-k\epsilon_0 V^2} dx = \frac{4Ld^*}{2dd^*-k\epsilon_0 V^2}. \tag{89}$$

Then, $W_{final} = \frac{1}{2}CV^2 < \frac{1}{2}\frac{4d^*LV^2}{2dd^*-k\epsilon_0 V^2}$ so that $\Delta W = W_{final} - W_{initial} < \underbrace{\frac{1}{2}\frac{4d^*LV^2}{2dd^*-k\epsilon_0 V^2} - \frac{\epsilon_0 LV^2}{d}}_{f(V)}$.

Remark 7.2. It is easy to verify that $f(V)$, for usual values of the parameters d, d^*, L, k and ϵ_0, and considering the range (71), is a continuous functions. In fact, if in $f(V)$, $2dd^* - k\epsilon_0 V^2 = 0$, we would obtain $V = \sqrt{\frac{2dd^*}{k\epsilon_0}} \approx 0.1503V$. But it easy to verify that this value is lower that $\inf(V_{min})_{inertia}$ (see (33)) so that $V \approx 0.1503V$ is not within the range of possible values for V.

To calculate the stationary points for $f(V)$, simply impose $\frac{df(V)}{dV} = \frac{4d^*Lk\epsilon_0 V^2}{(2dd^*-k\epsilon_0 V^2)^2} + \frac{4d^*L}{2dd^*-k\epsilon_0 V^2} - \frac{2\epsilon_0 L}{d} = 0$ from which, setting $V^2 = t$, we easily obtain $t = \frac{2dd^*(\epsilon_0 \pm \sqrt{\epsilon_0})}{k\epsilon_0^2}$ where $2dd^* > 0$, $k\epsilon_0^2 > 0$ and $\epsilon_0 - \sqrt{\epsilon_0} < 0$ achieving $\frac{2dd^*(\epsilon_0-\sqrt{\epsilon_0})}{k\epsilon_0^2} < 0$. Then, $V \in \mathbb{C}$, so that $t = \frac{2dd^*(\epsilon_0-\sqrt{\epsilon_0})}{k\epsilon_0^2}$ is to be discarded. Therefore

$$V = \sqrt{\frac{2dd^*(\epsilon_0+\sqrt{\epsilon_0})}{k\epsilon_0^2}} \tag{90}$$

We discard the negative root because this value of V would deform the membrane symmetrically with respect to the lower plate (condition physically impossible to achieve). It is easy to verify that V (see (90)) is a point of maximum for $f(V)$ (see Fig. 3a). Moreover, it is also easy to verify the inequality (88). □

7.2. *A Limitation for ΔW Obtained Starting from $|\mathbf{E}|$.*

Starting from models (1) and (4), we can write $\frac{\lambda^2}{(1-u(x))^2} = \frac{1}{\theta\lambda^2}\left(1+\left(\frac{du(x)}{dx}\right)^2\right)^3(1-u(x)-d^*)^2$ and taking into account that $\frac{\lambda^2}{(1-u(x))^2} = \theta|\mathbf{E}|^2$, we obtain $|\mathbf{E}|^2 = \frac{1}{\theta^2\lambda^2}\left(1+\left(\frac{du(x)}{dx}\right)^2\right)^3(1-u(x)-d^*)^2$. Therefore, taking into account both the initial potential energy and $W_{final} = \frac{1}{2}\epsilon_0|\mathbf{E}|^2$, we can write:

$$\Delta W = W_{final} - W_{initial} = \frac{\epsilon_0}{2\theta^2\lambda^2}\left(1+\left(\frac{du(x)}{dx}\right)^2\right)^3(1-u(x)-d^*)^2 - \frac{\epsilon_0 LV^2}{d} \tag{91}$$

And again, considering that $\frac{du(x)}{dx} < H$ and $1-u-d^* < 1-d^*$, equality (91) becomes:

$$\Delta W = W_{final} - W_{initial} \leq \frac{\epsilon_0}{2\theta^2\lambda^2}(1+H^2)^3(1-d^*)^2 - \epsilon_0 L\frac{V^2}{d}. \tag{92}$$

Proposition 7.2. *In* (83), $\epsilon_0 V^2 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\} < \frac{\epsilon_0}{2\theta^2\lambda^2}(1+H^2)^3(1-d^*)^2 - \epsilon_0 L \frac{V^2}{d}$ *in* (92).

Proof. If absurdly $\frac{\epsilon_0}{2\theta^2\lambda^2}(1+H^2)^3(1-d^*)^2 - \epsilon_0 L \frac{V^2}{d} \leq \epsilon_0 V^2 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\}$ and taking into account the condition (8), after simple algebraic calculations, we achieve $\frac{d^3 T (1+H^2)^3 (1-d^*)^2}{2\theta^2 \epsilon_0 L^3} < \frac{V^4}{d-d^*}$ that, in dimensionless conditions, becomes $V > \sqrt[4]{\frac{3.10 \cdot 10^3 T}{\theta^2}} 10^5$. In other words V should be greater than an extremely high amount. Then, it makes sense to write:

$$
\epsilon_0 V^2 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\} < \frac{\epsilon_0}{2\theta^2\lambda^2}(1+H^2)^3(1-d^*)^2 - \epsilon_0 L \frac{V^2}{d}. \tag{93}
$$
□

Finally, Proposition 7.2 help us to achieve an interesting increase for ΔW only depending on the electromechanical properties of the membrane (θ) and on how to fix the membrane to the lower plate. In fact, from (93) and (8), we can write $\Delta W \leq \epsilon_0 V^2 \Big\{ \frac{Ld^*}{d(d-d^*)} \Big\} < \frac{d^3 T}{2\theta^2 L^2 V^2}(1+H^2)^3(1-d^*)^2 - \frac{\epsilon_0 L V^2}{d}$ that, in dimensionless conditions, becomes $\Delta W < \underbrace{12\frac{T}{\theta^2 V^2} - V^2}_{g(V)}$.

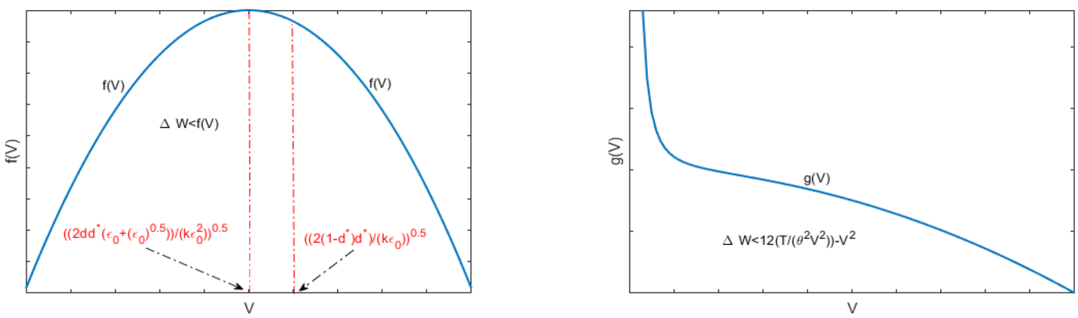


Figure 3. Area of possible values for ΔW according to (a) $\Delta W < f(V)$ and (b) $\Delta W < g(V)$.

8. Conclusions and Perspectives

The starting point of this work concerned the study of the stability of the equilibrium configurations of a second-order nonlinear two point boundary value problem relating to a 1D electrostatic membrane MEMS device. In this model, since **E** is locally orthogonal to the straight line tangent to the membrane at the point considered, $|\mathbf{E}|$ was considered proportional to the curvature of the membrane at the same point. The only equilibrium configuration obtained relates to the position of the membrane whereby the height reached in the middle is equal to the maximum allowed height (that is, an amount equal to the critical safety distance d^* from the upper plate).

Using the typical linearization technique of the mathematical model (adequately rewritten) around its equilibrium configuration, the study found the non-stability of the only equilibrium configuration obtained. It is worth pointing out that one of the most important causes of the instability of electrostatic membrane MEMS devices concerns the possibility that the p_{el} inside the device undergoes variations (albeit minimal) with respect to the expected values, also influencing the post-instability behavior.

As p_{el} depends on V, these possible fluctuations in the p_{el} can arise from fluctuations in V, so that attention was also paid to the evaluation of the maximum and minimum values of V. Specifically, the minimum value of V was obtained so that the membrane inertia is overcome ($(V_{min})_{inertia}$) and the maximum value of V so that the center line of the membrane does not exceed the critical safety distance ($(V_{max})_{admissible}$). This work is to be framed in studies dedicated to solving the inverse problem in a 1D electrostatic membrane MEMS device. In fact, starting from the deformation of the membrane, the range of permissible values of the applied V were evaluated.

The link between $(V_{min})_{inertia}$ and $(V_{max})_{admissible}$ was obtained and depended on the value of the mechanical tension T where the membrane was subjected. Thus, once the material constituting the membrane was chosen (which is equivalent to fixing T) we obtained $\frac{(V_{min})_{inertia}}{(V_{max})_{permissible}}$. We observed that the mathematical models adhering to the physical reality of MEMS device were surprisingly complex and did not facilitate easy study. Some simplifications in the geometry of the device were necessary in order to obtain a simpler mathematical model to study.

Furthermore, some considerations on the variation of potential energy have made it possible to identify optimal control conditions also starting from $|\mathbf{E}|$.

The results obtained from the study of the simplified model hardly matched with the experimental data but will provide interesting qualitative indications on the behavior of these devices characterized by simplified geometries. If, on the one hand, the extension to 2D cases of this study is desirable, on the other hand, the future choice of less simplified geometries will make the approach presented here more robust to deal with inverse problems related to the electrostatic MEMS membrane.

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