Mask-Constrained Power Synthesis of Linear Arrays with Even Excitations

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T. Isernia and A.F. Morabito

***Abstract*—An existing effective approach for the synthesis of linear arrays providing power patterns lying in arbitrary masks is extended to the case where an even distribution is required on element excitations. The approach consists of a linear programming optimization plus a polynomial factorization, and it allows one to deal with a considerably reduced computational burden with respect to the previous approaches. It is shown that such a circumstance allows one to successfully deal with problems that cannot be solved by means of the previous techniques. An extension of the overall strategy to the synthesis of fields guaranteeing optimal performances in terms of directivity or ripple in the main-beam region is also given.**

***Index Terms*—Antenna theory, arrays, power synthesis.**

I. INTRODUCTION

The synthesis of the complex excitations of a fixed-geometry linear array able to fulfill given lower and upper bounds on the radiated power distribution is an important topic in Antenna community (see [1]-[7] and references therein). For the case of linear equispaced arrays, a powerful technique allowing one to effectively solve the problem by means of a Linear Programming (LP) procedure followed by a polynomial factorization has been introduced in [3]. Such a framework also allows one to identify the minimal number of elements required to fulfill the assigned specifications as well as optimal "reference" solutions exploitable in the synthesis of reconfigurable [4] and sparse isophoric arrays [5].

In some applications, one is interested not only in fulfilling the given bounds on the power density, but also in achieving an *even* complex pattern, or an *even* excitations distribution. Such a requirement is of interest, for example, in case when one wants to produce a reconfiguration from a shaped to a difference pattern, where the symmetry of the beam forming network will enable a very simple and accurate reconfiguration [4]. Also, such a capability is necessary for the optimal synthesis of shaped beams having a circular symmetry [6].

Next, we show how to extend the theory and procedures introduced in [1] and discussed in [3] to such a case. The proposed formulation is completely different and more convenient with respect to the one developed in [6]. In fact, we are able to deal with polynomials whose order is halved. Results below are also of interest in the 'phase-retrieval' problem which occurs in antenna diagnostics [8],[9], signal processing [10], optics, crystallography [11], and many other branches of applied science.

In Section II we review the results in [3]. In Section III we develop the extension to the case where one looks for minimal-dimension uniformly-spaced arrays having even excitations. Then, in Section IV we compare the proposed approach with the previous ones, and in Section V we generalize the synthesis procedure to the cases where one looks for maximum directivity or minimum ripple performances within the main-beam region. Conclusions follow.

II. SUMMARY OF PREVIOUS RESULTS

Let us consider a linear array composed of N isotropic antennas located along the x-axis with a constant inter-element spacing equal to d. Moreover, let us denote by λ and θ the wavelength and the angle between the array axis and the observation direction, respectively. Then, by defining $u=\beta d\cos\theta$ and $\beta=2\pi/\lambda$, according to the results in [3], the square-amplitude far field of such a radiating system can be expressed as a linear combination of $2N$-1 complex coefficients, say $D_{-N+1},\ldots,D_0,\ldots,D_{N-1}$, i.e.,

$$P(u)=\sum_{n=-N+1}^{N-1} D_n e^{jnu} \tag{1}$$

By straightforward derivations, expression (1) can also be analyzed as the restriction to the unit circle of a polynomial in the complex variable $z=e^{ju}$ [3]. As $P(u)$ must be a real and non-negative function, one also has:

$$D_n=D_{-n}^{*} \qquad n=1,\ldots,N-1 \tag{2}$$

$$\sum_{n=-N+1}^{N-1} D_n e^{jnu} \geq 0 \tag{3}$$

* meaning complex conjugation. Under such assumptions, the synthesis of a power pattern distribution lying in an arbitrary mask can be performed by solving the following system of linear inequalities in the unknowns D_n, i.e.,

$$\begin{cases} 0 \leq LB(u) \leq \displaystyle\sum_{n=-N+1}^{N-1} D_n e^{jnu} \leq UB(u) & \text{(4.a)} \\ D_n = D_{-n}^{*} \qquad n=1,\ldots,N-1 & \text{(4.b)} \end{cases}$$

where UB and LB respectively denote the upper and lower bounds of the assigned mask. In enforcing them, it is appropriate to scan the u variable through the whole range $[-\pi, \pi]$. In fact, by so doing, it is possible to keep under control the power pattern into its whole periodicity range as well as to avoid *superdirective* solutions [2] in case $d<\lambda/2$.

Since the band-limitedness of $P(u)$ allows a sufficiently fine discretization of the above constraints, the optimization problem (4) constitutes a *LP problem* [3]. Once a solution to system (4) is found, the element excitations can be retrieved by factorizing (1) into two complex-conjugate factors, i.e.,

$$P(u)=F(u)F^{*}(u) \tag{5}$$

so that F (or F^*) can be then interpreted as the array factor [3].

Notably, factorization (5) is not unique, as a number of equivalent array factors, i.e., excitation sets, all generating the same power pattern $P(u)$ can be identified through a simple "zero-flipping" operation [3],[4]. In particular, the number of equivalent solutions results equal to $2^{\upsilon/2}$, υ being the number of *complex* zeros of $P(z)$ not lying on the unit circle. This property

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may be profitably exploited in a number of applications of actual interest, including the synthesis of phase-only reconfigurable arrays [4].

The above tools can obviously be used to solve all those *synthesis* and *phase-retrieval* problems where the signal of interest can be reduced to one-dimensional polynomials. These include a number of cases exhibiting particular symmetries [12], two-dimensional u-v factorable patterns, as well as the synthesis of sources enclosed in a circular cylinder of fixed radius and invariant along the axis direction [3]. In the latter case, one can exploit a truncated expansion into cylindrical harmonics in order to find all the different fields satisfying the design constraints, and then synthesize the corresponding sources by means of a linear inversion. Notably, this approach can be extended to sources invariant along one direction and enclosed in a convex hull and to near-field specifications by using the so-called "reduced radiated field" concept [7]. Obviously, the aim and constraints in [7] are distinct from the ones in the remainder of the present contribution, as [7] is not concerned with evenness requirements. However, the two approaches can be eventually integrated in order to enlarge the range of applicability of the proposed synthesis procedure.

III. SYNTHESIS OF EVEN EXCITATIONS AND FIELDS

A necessary condition to achieve even field distributions is that the power pattern distribution is even as well. As a consequence, representation (1) can be simplified as:

$$P(u) = D_0 + \sum_{n=1}^{N-1} D'_n \cos(nu) \tag{6}$$

where coefficients $D'_n=2D_n$ are real since $P(u)$ is a real function.

Notably, exploitation of representation (6) does not guarantee that the *field* pattern is even as well. In fact, one can still have an anti-symmetric part in the excitations (and in the field distribution) which still results in an even function when considering its square-amplitude behavior.

Therefore, it is of interest to understand whether one can have a formulation of the synthesis problem analogous to the one above while also guaranteeing the evenness of the excitations. To this end, let us start by noting that the term $\cos(nu)$ can be expressed as a polynomial of order n in the $\cos(u)$ variable [13]. Hence, a further possible expression for $P(u)$ guaranteeing evenness of the latter is given by

$$P(u) = \sum_{n=0}^{N-1} D''_n \cos^n(u) \tag{7}$$

where, for any fixed value of N, coefficients D''_n are known linear functions of coefficients D_n. As (7) is an one-dimensional polynomial in the auxiliary variable $\zeta=\cos(u)$, the Fundamental Theorem of Algebra guarantees that it can be expressed in terms of its zeros as follows:

$$P(u) = D''_{N-1} \prod_{n=1}^{N-1} \left[\cos(u) - \zeta_n\right] \tag{8}$$

where the n-th root ζ_n can be either purely real or complex. As far as the complex zeros of (8) are concerned, because of the fact that $P(u)$ has to be a *real* function, they must appear in complex conjugate couples, so that if $(\xi_n+j\eta_n)$, with $\eta_n\neq0$, is a root, then $(\xi_n-j\eta_n)$ must be a root as well. As a consequence, one can split the complex zeros in two different groups where if ζ_n belongs to the first group then ζ_n^* belongs to the second one. Also note that if ζ_n is a root with a multiplicity different from 1 then ζ_n^* is a root with the same multiplicity.

Then, as long as (N-1) is even and quantities ζ_n in representation (8) are not real, $P(u)$ can also be written as:

$$P(u) = \sqrt{D''_{N-1}} \prod_{n=1}^{\frac{N-1}{2}} \left[\cos(u) - \zeta_n\right] \sqrt{D''_{N-1}} \prod_{n=1}^{\frac{N-1}{2}} \left[\cos(u) - \zeta_n^*\right] \quad (N\ odd) \tag{9}$$

In such a case, the two factors constituting (9), which are indeed both even, can be interpreted as the complex conjugate of each other, and hence they can be seen as the array factor and its complex conjugate.

Unfortunately, nothing ensures that all roots of (8) are complex, and the possible occurrence and multiplicity of real roots has to be discussed as well. Obviously, this is also needed in case of an even number of antennas, i.e., where (N-1) is odd, and a real root is necessarily present.

Simple arguments show that, apart from the possible value $\zeta_n=R_n=-1$, in order to guarantee that (9) is the power pattern associated to an even array factor, the real roots of (8) must have an even multiplicity. In fact, a factor of the kind $\cos(u)-R_n$ can be written as:

$$\begin{aligned} \cos(u) - R_n &= 1 + \cos(u) - (R_n + 1) = \\ &= \left[\sqrt{2}\cos\left(\frac{u}{2}\right) - \sqrt{R_n+1}\right]\left[\sqrt{2}\cos\left(\frac{u}{2}\right) + \sqrt{R_n+1}\right] \end{aligned} \tag{10}$$

Therefore, the unique real root R_n having a single (or odd) multiplicity able to give rise to two factors contemporarily being even and complex conjugate of each other is the case R_n=-1, where the two factors in (10) become identical. Notably, this is the case which indeed occurs when (N-1) is odd, i.e., when the number of elements in the array is even. In such a case, the term $\cos(u/2)$ is associated with the known circumstance that (in case of symmetrical excitations) the array factor can be expressed in terms of odd powers of $\cos(u/2)$ [14], and the power pattern can be written as:

$$P(u) = \tau \cos\left(\frac{u}{2}\right) \prod_{n=1}^{\frac{N-2}{2}} \left[\cos(u) - \zeta_n\right] \tau \cos\left(\frac{u}{2}\right) \prod_{n=1}^{\frac{N-2}{2}} \left[\cos(u) - \zeta_n^*\right] \quad (N\ even) \tag{11}$$

where τ is a real constant. In such a case, the two factors constituting $P(u)$ are both even and can also be interpreted as the array factor and its complex conjugate. As in (9), the factorization is not unique, as one can interchange the role of ζ_h and ζ_h^* $(\forall h)$ and this will produce a new solution (in terms of field) as long as ζ_h is not real.

Independently from the evenness or oddness of N, the number of equivalent array factors, i.e., excitation sets, corresponding to the sought power pattern will be equal to $2^{\alpha/2}$, α denoting the number of complex, i.e., *non-real*, zeros of $P(u)$. As a consequence, the additional condition that is needed for factorization into even and complex-conjugate factors is given in the case at hand as:

$$\sum_{n=0}^{N-1} D''_n \zeta^n \geq 0 \qquad \forall \operatorname{Re}(\zeta) \tag{12.a}$$

as long as N is odd, or

$$\begin{cases} \sum_{n=0}^{N-1} D''_n \zeta^n \geq 0 & \forall \operatorname{Re}(\zeta) \geq -1 \\ \sum_{n=0}^{N-1} D''_n \zeta^n \leq 0 & \forall \operatorname{Re}(\zeta) < -1 \end{cases} \quad (12.b)$$

as long as N is even. In fact, while conditions (4) already guarantee that complex roots appear in complex-conjugate pairs, conditions (12) entail that any possible *real* root of $P(u)$ exhibits an even multiplicity, and that a further zero equal to -1 is present if N is even.

Unfortunately, because of the fact that one should sample the entire real axis of the ζ plane, conditions (12) cannot be used in practice, so that alternative equivalent conditions are needed. In this respect, it is useful to note that in the interval $-1 \leq \operatorname{Re}(\zeta) \leq 1$, conditions (12) are fully equivalent to require that $P(u)$ is a non-negative function in the interval $-\pi \leq u \leq \pi$ and hence they are already enforced through constraint (4.a) or by an equivalent condition on the pattern in terms of the D''_n coefficients. Therefore, provided this condition is enforced, one just needs to consider the intervals $|\operatorname{Re}(\zeta)|>1$. This can be done, in case N is odd, by adding the following constraint:

$$\sum_{n=0}^{N-1} D''_n \left(\frac{1}{\rho}\right)^n \geq 0 \quad (13)$$

ρ being a *real* variable spanning the interval (-1,1). Obviously, an analogous strategy is possible in case N is even.

By taking these results into account, the synthesis of a power pattern lying in an arbitrary mask and being factorable into two *even* factors that are complex conjugates of each other can be performed by solving in the unknowns $D''_0,\ldots,D''_{N-1}$ the following problem:

$$\begin{cases} 0 \leq LB(u) \leq \sum_{n=0}^{N-1} D''_n \cos^n(u) \leq UB(u) \quad u \in [-\beta d, \beta d] & (14.a) \\ \rho^{1-N} \sum_{n=0}^{N-1} D''_n \rho^{N-1-n} \geq 0 \quad \rho \in [-1,1] & (14.b) \end{cases}$$

and performing a subsequent factorization. In fact, condition (14.a) takes into account conditions (4.a) and (4.b), as well as conditions (12) for the interval $|\operatorname{Re}(\zeta)|\leq 1$, whereas (14.b) is equivalent to condition (13) (and to the analogous one in case N is even) and allows one to reason in terms of a polynomial of order (N-1) once the sign of the factor ρ^{1-N} is taken into account. In such a way, one is also able to avoid singularities in the actual implementation. Moreover, when considering $\cos(u)=-1$ and $\rho=-1$, intersection of conditions (14.a) and (14.b) allows guaranteeing the occurrence of a zero with odd multiplicity in $\zeta=-1$ in case N is even, as required by (11) and (12.b).

As in problem (4), provided that a sufficiently fine discretization is adopted, problem (14) can be solved as a LP problem (where some additional condition or requirement on the power pattern can eventually be added, as shown in Section V). Once (14) is solved, (7) can be factorized though either (9) or (11) and the array excitations can be retrieved through a fast and easy operation, e.g., by performing the inverse Fast Fourier Transform (FFT) of the achieved array factors.

The synthesis procedure is hence divided into two steps:

- a feasibility criterion allowing to ascertain *a priori*, i.e., before actual synthesis, whether or not the given shaped beam can be realized with a given number of antennas;
- the determination of the actual array excitations.

By separating the assessment of the given mask's feasibility from the array actual synthesis, the approach allows avoiding computationally heavy 'trial-and-error' procedures. Moreover, the extremely low computational burden of problem (14) allows a simple and fast way to find the size of the *minimally redundant* array fulfilling the assigned requirements. In fact, for an assigned power mask and a given d value, the minimum feasible number of array elements required to generate the desired pattern can be identified by iteratively solving problem (14) for decreasing values of N until constraints become so strict to prevent the existence of a solution.

Both steps of the synthesis procedure imply the solution of canonical problems for which a huge number of numerical routines are available. However, the second step of the procedure implies the factorization of a polynomial of large order, which induces some practical limitations when using commercially available routines. In fact, whatever the numerical tool the user exploits, such a number strongly affects the factorization accuracy [15]. As a consequence, limitations are induced on the maximum allowed array size which can be dealt with by methods based on factorization.

In this respect, the proposed approach allows a considerable gain with respect to previous techniques [1],[6]. In fact, while in the latter the number of involved zeros was equal to (2N-2) {see (1) and [1],[3],[6]}, in the above formulation we only need to deal with polynomials of order (N-1). Therefore, for a given maximum number of zeros allowed by the calculator at hand, we are now able to *double* the maximum allowed size of the source. This chance may be very useful in several applications, e.g., Compressive-Sensing-based array design procedures. In fact, a very small inter-element spacing, i.e., a very large value of N, is usually adopted in the latter [16].

Finally, it has to be noted that the most common numerical computing environments find the polynomial roots by determining the eigenvalues of the so-called "companion matrix" [17]. Resources required by this operation are of the order Ω^3 (time) and Ω^2 (storage), Ω denoting the polynomial degree [18]. Therefore, the new procedure is *eight times* faster and requires just *a quarter* of storage resources with respect to the previous approaches [1],[6]. An example highlighting the advantages of the approach with respect to the state-of-the-art methods in terms of the factorization's accuracy and computational weight is shown in the following Section.

IV. An Application to Circularly-Symmetric Sources for Satellite Communications

To exemplify both the interest of the procedure as well as its additional capabilities with respect to previous approaches, we compared the different techniques in the synthesis of patterns of interest in satellite applications. In particular, we considered two recent Invitations To Tenders (ITTs) of the European Space Agency (ESA) [19],[20] concerning the synthesis of sparse ring arrays for the full Earth coverage from a GEO satellite. The required antenna must be of the kind designed in

[21],[22], i.e., isophoric and reconfigurable at the same time. In particular, steerable beams switchable between the two different widths shown in Fig. 1 have to be generated by exploiting a circular aperture having a maximum diameter of 120λ. As shown in [5],[6],[21]-[23], a convenient way to solve such a kind of problem is to obtain the array layout as the discretization of a reference continuous source radiating a circularly-symmetric far-field pattern fulfilling "at best" the technical requirements. As discussed in [6], a slice of this pattern can be in turn synthesized by exploiting a "virtual" equispaced linear array having a size equal to the continuous source's diameter and an even excitations distribution.

With reference to the 'large beam' configuration, we solved such a problem by using formulation (14), selecting N=241, d=λ/2, and the upper and lower bounds as described in [20]. In particular, constraints required to realize on the power pattern a separation of at least 20 dB between the minimum value attained in the range $|\theta'|\leq 1.625°$ and the maximum value attained for $|\theta'|\geq 3.795°$, with $\theta' = \theta + 90°$.

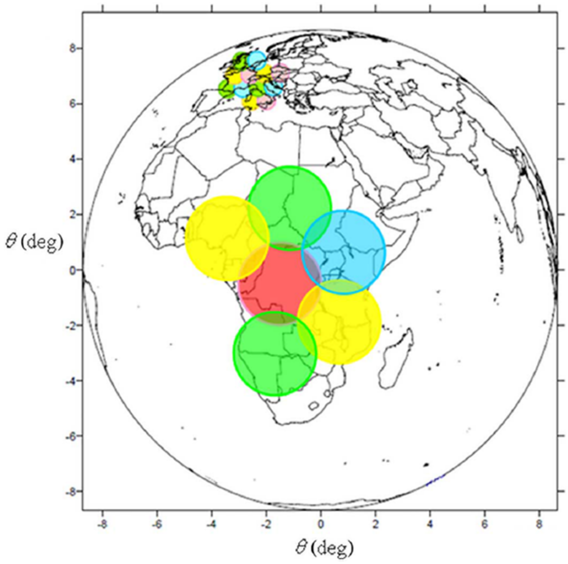


Fig. 1. Full Earth coverage from a GEO satellite by means of multiple steerable and "zoomable" beams (from European Space Agency ITTs [20]).

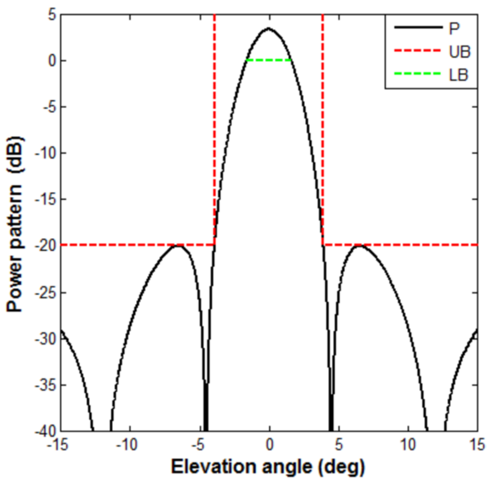
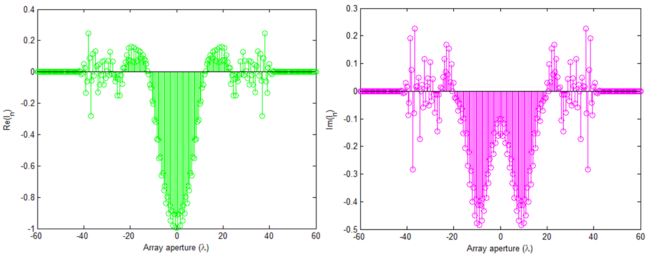


Fig. 2. First test case – ESA ITTs [20]: synthesized power pattern (black color) and adopted upper (red color) and lower (green color) bound functions.



(a) (b)

Fig. 3. Real (a) and imaginary (b) parts of an even array excitations set generating the power pattern depicted in Fig. 2.

The simulation has been performed by a calculator equipped with an Intel Core i7-3537U 2.50 GHz CPU and a 10 GB RAM. Despite the extremely high polynomial degree, i.e., Ω=240, the overall synthesis (including excitations determination) required just 113 seconds and no issues arose during the factorization step. The synthesized power pattern and the adopted mask are shown in Fig. 2, while one amongst the different possible sets of excitations for the auxiliary linear array is given in Fig. 3. Notably, the approach is able to find all the $2^{\Omega/2}$ possible solutions, and such a multiplicity can be exploited to effectively realize the required phase-only reconfiguration along the guidelines in [4].

To assess the computational advantages of the proposed approach, we also tried to solve the problem through the procedure in [6]. In so doing, all parameters common to both approaches (upper and lower bound functions, number of sampling points in the u and ρ variables, array geometry, and so forth) have been kept constant, while the remaining ones (about the directivity and Q-factor issues dealt with in [6]) have been deactivated. Notably, the LP step alone required 245 seconds, while the large number of polynomial roots, i.e., 480, made an accurate factorization impossible. Hence, the theoretical advantages discussed in Section III are fully confirmed.

V. Further Capabilities of the Basic Approach and a Couple of Applications

The basic step (14), i.e., the one preceding factorization, can be eventually complemented by some additional constraints and/or cost functional to be optimized.

As long as such an additional constraint or performance parameter is linear in terms of the auxiliary variables D''_p, the problem remains of the LP class. A possible requirement that still allows one to deal with LP problems is the minimization of the *overall radiated power*, in such a way to improve the directivity in the shaped-beam zone. In fact, the overall radiated power linearly depends on the auxiliary variables.

Interestingly, one can also consider a wider class of constraints and/or requirements. In fact, simplicity and global optimality of the approach still hold true as long as the additional constraints or the objective function are convex (rather than linear, which is a more strict constraint) in terms of the auxiliary unknowns. In these cases, one will have to consider a Convex Programming (CP) problem (which still admits a unique optimal value of the objective function if any) instead of a LP one. Notably, effective off-the-shelf numerical routines exist for both LP and CP problems [24]. These circumstances further enlarge the range of applicability of the presented approach, making it very flexible and versatile.

To show the advantages deriving from such a feature, we consider in the following two synthesis problems having two different objective functions. In particular, we deal with the maximization of the directivity and the minimization of the ripple in the main-beam region of a shaped power pattern lying in a fixed mask. The outcomes of the two test cases are separately discussed in the following two subsections, which share the array geometry, i.e., N=50 and d=λ/2, the number of sampling points in the u variable, i.e., 601, and the adopted mask, i.e., a ripple not larger than ±1 dB for $|u|\leq 0.7$ and a sidelobe level not larger than -20 dB for $|u|\geq 1$. The mask considered in both examples is shown in Fig. 4. As in the previous test case, all simulations have been performed by a calculator equipped with an Intel Core i7-3537U 2.50 GHz CPU and a 10 GB RAM. This time, each example (including identification of excitations) required less than 10 seconds to be generated.

V.1 Synthesis of even fields guaranteeing the maximum average directivity in the shaped zone

To maximize the directivity within the main beam, one can minimize, under constraints (14), the following objective function:

$$\Theta\left(D''_0,\ldots,D''_{N-1}\right)=\frac{\Pi_{visible}}{\Pi_{main\ beam}} \tag{15}$$

where

$$\Pi_{visible}=\int_0^{\pi} P(\theta)\sin\theta d\theta \tag{16}$$

$$\Pi_{main\,beam}=\int_{\arccos\left(\frac{\psi}{\beta d}\right)}^{\arccos\left(\frac{-\psi}{\beta d}\right)} P(\theta)\sin\theta d\theta \tag{17}$$

respectively represent (but for a common constant factor) the overall radiated power and the power transmitted in the range $u\epsilon[-\psi,\psi]$ identifying the shaped-beam region (with $0<\psi<\beta d$). Notably, both expressions (16) and (17) are real non-negative linear functions of the auxiliary unknowns. Therefore, the minimization of (15) under conditions (14) can be dealt with as a Linear-Fractional Programming (LFP) problem [25].

Notably, positivity of (17) allows, by exploiting the Charnes-Cooper transformation [25] and introducing an auxiliary variable t, to recast the synthesis as the following LP problem in the auxiliary unknowns $t, Y_0,\ldots,Y_{N-1}$:

minimize

$$\Phi\left(Y_0,\ldots,Y_{N-1}\right)=-\int_{\arccos\left(\frac{\psi}{\beta d}\right)}^{\arccos\left(\frac{-\psi}{\beta d}\right)}\left[\sum_{n=0}^{N-1} Y_n\cos^n\left(\beta d\cos\theta\right)\right]\sin\theta d\theta \tag{18}$$

subject to

$$\begin{cases} 0\leq LB(u)t\leq \sum_{n=0}^{N-1} Y_n\cos^n(u)\leq UB(u)t & \text{(19.a)}\\ \rho^{1-N}\sum_{n=0}^{N-1} Y_n\rho^{N-1-n}\geq 0 & \text{(19.b)}\\ t\geq 0 & \text{(19.c)}\\ \int_0^{\pi}\left[\sum_{n=0}^{N-1} Y_n\cos^n\left(\beta d\cos\theta\right)\right]\sin\theta d\theta=1 & \text{(19.d)}\end{cases}$$

where

$$Y_n=tD''_n \quad n=0,\ldots,N-1 \tag{20}$$

By solving such a problem for ψ=0.7, we achieved t=1.39 and the power pattern depicted in Fig. 4. In the range [-ψ,ψ], the ripple turned out being lower than ±0.93 dB while the minimum, average, and maximum directivity resulted equal to 4.58 dB, 6.10 dB, and 6.44 dB, respectively. As it can be seen, all constraints are fulfilled. Furthermore, the maximum directivity is just 0.12 dB lower than the directivity pertaining to a theoretical power pattern being constant in the region [-ψ,ψ] and zero elsewhere. Also note that the field amplitude keeps quite close to the maximum value but for the angles approaching ψ and -ψ. Fig. 5 shows one of the (even) array excitation sets corresponding to the synthesized power pattern.

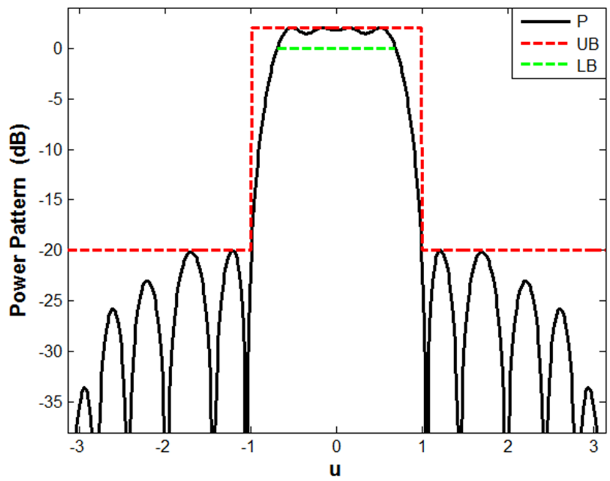
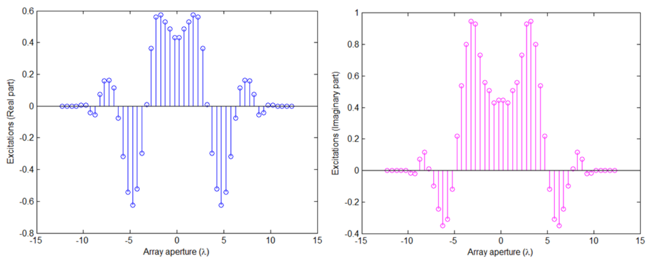


Fig. 4. Second test case – Maximum directivity: synthesized power pattern (black color) and adopted upper (red color) and lower (green color) bounds.



(a) (b)

Fig. 5. Real (a) and imaginary (b) parts of an even array excitations set generating the power pattern depicted in Fig. 4.

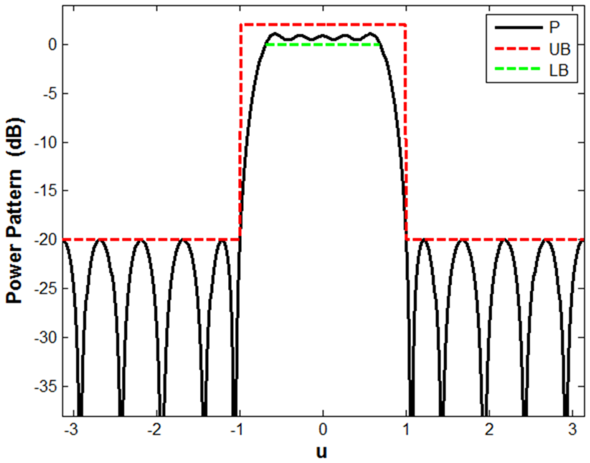
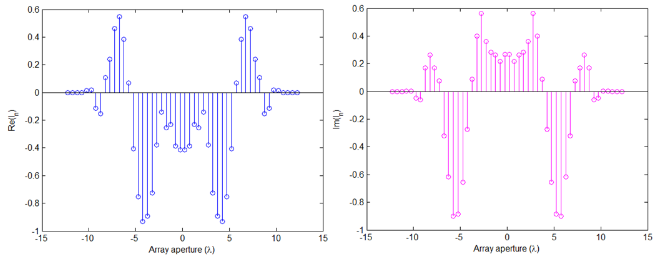


Fig. 6. Third test case – Minimum ripple: synthesized power pattern (black color) and adopted upper (red color) and lower (green color) bound functions.



(a) (b)

Fig. 7. Real (a) and imaginary (b) parts of an even array excitations set generating the power pattern depicted in Fig. 6.

V.2 Synthesis of even fields guaranteeing the minimum ripple

To minimize the main beam's ripple, we minimized, under constraints (14), the *variance* exhibited by the power pattern in the region of interest, i.e.:

$$\Omega\left(D''_0,\ldots,D''_{N-1}\right)=\frac{1}{2\psi}\int_{-\psi}^{\psi}\left[P(u)-\overline{P}\right]^2 du \tag{21}$$

denoting with [-ψ,ψ] the shaped-beam zone and with

$$\overline{P}=\frac{1}{2\psi}\int_{-\psi}^{\psi}P(u)du \tag{22}$$

the average value achieved by $P(u)$ inside it. Note that the cost function is a positive-definite quadratic function of the auxiliary unknowns D''_p, so that the minimization of (21) subject to constraints (14) is a CP problem.

By using again ψ=0.7, we achieved a power pattern exhibiting in the shaped-beam zone a ripple not larger than ±0.48 dB. Inside such a region, the minimum, average, and maximum directivity result are equal to 5.35 dB, 5.94 dB, and 6.3 dB, respectively. The synthesized power pattern is shown in Fig. 6. As in the previous test case, all the enforced constraints are satisfied. The maximum directivity is 0.14 dB lower than the one achieved in the previous test case and 0.26 dB lower than the theoretical bound introduced with reference to the previous example. One of the even array excitation sets generating the synthesized power pattern is depicted in Fig. 7. As expected, one has some trade-off amongst directivity and ripple performances.

VI. CONCLUSIONS

A powerful approach available in the power synthesis of array antennas radiating shaped beams lying in an arbitrary mask has been extended and optimized for the case where excitations must be even. The resulting design procedure keeps the same effective structure of its predecessors (as it is still reduced to a LP problem plus a polynomial factorization) but it allows one to deal with polynomials having half the order. Therefore, it is much faster and accurate in cases where both approaches can be used, and it can deal with sources having twice the size of the ones manageable by previous approaches. The expected advantages have been confirmed through an example concerning a computationally hard synthesis problem.

As an additional contribution, it has been shown the capability of the approach to deal with additional requirements such as minimum-ripple or maximum-directivity performances in the shaped-beam region. In fact, the basic approach can be modified (taking advantage from LFP and CP, respectively) in such a way that globally optimal solutions in terms of source size or radiation performance are still guaranteed.

The overall results considerably extend the capabilities of the synthesis approach, which may now be exploited in a larger set of applications.

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