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A new Lyapunov stability analysis of fractional-order systems with nonsingular kernel derivative

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Abstract This study introduces a new and promising stability approach for Caputo-Fabrizio (CF)-fractional-order system. A new fractional comparison principle for this nonsingular kernel fractional derivative is proposed. Next, a key inequality is suggested to analysis the Lyapunov-based stability of assumed systems. Afterwards, class-K functions are established to analysis of fractional Lyapunov direct method. At last, an explanatory example is given to validate the proposed idea. This new and novel approach can be expanded to the other types of nonsingular kernel derivatives due to a simple and effective idea beyond the proposed procedure.

1. Introduction

These days, global excitement for fractional calculus (FC) has been seemingly exponential. Due to the memory property of fractional derivatives, FC has attained much attention for modeling of image processes, applied mathematics, physics, and engineering [1–15]. However, in the old definitions of the fractional derivatives such as Riemann–Liouville, Caputo type, Grünwald-Letnikov derivative and so on, we face with some difficulties and the non-existence of a number of significant properties that make the fractional mathematical modeling not enough effective. The most considerable problem is the singularity of the kernel existing in their definitions that makes FC as an expansive subject field. Due to this matter, a number of researches was implemented to develop new fractional derivative to overwhelm some of the defects in the aforesaid derivatives [16–23].

As we have stated above, there was mush efforts to overcome the problem related to the singular kernel of the fractional derivatives, Caputo and Fabrizio tackled this problem effectively by introducing a different derivative with non-singular kernel [24] which was followed up by Atangana and Baleanu to propose a generalization concept of the Caputo-Fabrizio (CF) derivative with exponential kernel [25]. By introducing the non-singular kernel fractional

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derivatives in the literature, enormous researches have been devoted to explorer the undiscovered parts of these significant derivatives from the theoretical, numerical and modeling approaches [26–32].

In nonlinear systems, Lyapunov's direct method (additionally called the Lyapunov's second technique) outfits a way to examination the stability of a system implicitly solving the differential equations. The technique sums up the thought, which portrays the system is stable if there exist some Lyapunov function prospects for the system. The Lyapunov direct method has an adequate condition to demonstrate the stability of nonlinear systems, which implies the system may still be stable, regardless of whether one cannot discover a Lyapunov function prospect to infer the system stability characteristic [33,34]. In recent years, FC was acquainted to the stability analysis of nonlinear systems [35–39], such that integer-order schemes of stability analysis were formulated to fractional-order dynamic systems.

Motivated by aforementioned findings and for expanding the use of FC in nonlinear systems, we introduce exponential and Lyapunov stability for CF-fractional-order systems with a view to enhance the information of both system theory and FC. At the same time, the reality that calculation turns out to be quicker and memory winds up less expensive makes the utilization of this new fractional-order system, in fact, feasible and affordable. In this regards, we firstly establish a fractional order differential inequality. This inequality provides a new result regarding the Lyapunov-based stability for fractional-order system with CF-derivative. Second, a fractional comparison principle for the CF fractional derivative is developed. Using this principle for class-k functions, the stabilization of fractional nonautonomous systems are discussed, the similar approach was proposed for Caputo derivative system in [40,41]. In fact, we establish the definition of exponential stability and the Lyapunov direct method for CF-fractional-order systems. An illustrative model is given to exhibit the effectiveness of the proposed scheme.

The paper is sorted out as follows. In Section 2, the main definitions concerning CF-fractional derivatives given. By introducing a significant inequality for CF-fractional derivative, exponential stability of the fractional-order system is discussed in Section 3. Besides, the Lyapunov-based stability of CF-fractional-order systems is also developed and by considering the definition of class-k functions, fractional comparison principle and the stabilization of the non-autonomous fractional order system are discussed in details. A case study is addressed in Section 4. Finally, a number of conclusion remarks are provided in Section 6.

2. Preliminaries

In this section, the CF-derivative with its anti-derivatives are recalled [24,31].

Definition 2.1. Suppose that $\mathcal{Z}(v) \in \mathbf{H}^1(r,s),\ s > r,\ \zeta > 0,\ \zeta \in [0,1]$, the non-integer derivative introduced by Caputo and Fabrizio [24] is formulated as:

$$ {}^{\mathcal{CF}}\mathfrak{D}^{\zeta}\mathcal{Z}_v(v) = \frac{M(\zeta)}{1-\zeta}\int_r^v \mathcal{Z}'(\tau)\exp\left[-\zeta\frac{v-\tau}{1-\zeta}\right]d\tau, \tag{2.1} $$

in which $M(\zeta)$ is a normalization function. However, if $\mathcal{Z}(v) \notin \mathbf{H}^1(r,s)$ then, the statement (2.1) can be expressed as

$$ {}^{\mathcal{CF}}\mathfrak{D}^{\zeta}\mathcal{Z}_v(v) = \frac{\zeta M(\zeta)}{1-\zeta}\int_r^v (\mathcal{Z}(v) - \mathcal{Z}(\tau)) \times \exp\left[-\zeta\frac{v-\tau}{1-\zeta}\right]d\tau. \tag{2.2} $$

Definition 2.2. Assume that $0 < \zeta < 1$. The left non-integer integral of order ζ of a function $\mathcal{Z}(v)$ is stated by

$$ {}^{\mathcal{CF}}I^{\zeta}_{0^+}(\mathcal{Z}(v)) = \frac{(1-\zeta)}{M(\zeta)}\mathcal{Z}(v) + \frac{\zeta}{M(\zeta)}\int_0^v \mathcal{Z}(\tau)d\tau, \quad v \geqslant 0. \tag{2.3} $$

Definition 2.3. Suppose that $0 < \zeta < 1$. Laplace Transform of CF-derivative of a function $\mathcal{Z}$ is described as:

$$ L[{}^{\mathcal{CF}}\mathfrak{D}^{\zeta}\mathcal{Z}(v)](\tau) = \frac{(2-\zeta)M(\zeta)}{2(\tau+\zeta(1-\tau))}(\tau L[\mathcal{Z}(v)](\tau) - \mathcal{Z}(0)), \quad \tau > 0, \tag{2.4} $$

in which $L[\mathcal{Z}(v)]$ stands for the Laplace Transform of function $\mathcal{Z}$.

Proposition 2.1. Let $0 < \zeta < 1$, then [42]:

$$ {}^{\mathcal{CF}}I^{\zeta}({}^{\mathcal{CF}}\mathfrak{D}^{\zeta}(X(t))) = X(t) - X(0) \tag{2.6} $$

3. Main results

Using the CF-derivative, we take into consideration the following noninteger-order system:

$$ \begin{cases} {}^{\mathcal{CF}}\mathfrak{D}^{\zeta}z(v) = g(z,v) \\ z(v_0) = v_0, \quad \zeta \in (0,1) \end{cases} \tag{3.6} $$

where $z \in \mathbb{R}^n$ is the system state, $v_0 \in \mathbb{R}^n$ is the initial state, $g : [v_0,\infty) \times \Omega \to \mathbb{R}^n$ is piecewise continuous function in v and locally Lipschitz in $z \in [v_0,\infty) \times \Omega$and $\Omega \in \mathbb{R}^n$ is a domain including the origin $z = 0$. If $g(0,v) = 0$ then $z = 0$ is called the equilibrium point of the system (3.6).

Lemma 3.1. *Suppose that $z = 0$ is an equilibrium point for the system (3.6), and $D \subset \mathbb{R}^n$ be a domain including the origin. Assume $\mathcal{V}(v,z(v)) : [0,\infty) \times D \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz w.r.t. z such that*

$$ \zeta_1||z||^c \leqslant \mathcal{V}(v,z(v)) \leqslant \zeta_2||z||^{cd} \tag{3.7} $$

$$ {}^{\mathcal{CF}}_{0}\mathfrak{D}^{\zeta}_{v}\mathcal{V}(v,z(v)) \leqslant -\zeta_3||z||^{cd} \tag{3.8} $$

in which $v \geqslant 0, z \in D, \zeta \in (0,1), \zeta_1, \zeta_2, \zeta_3, c$ and d are arbitrary positive constants. Then $z = 0$ is exponential stable. If the hypotheses hold globally on $\mathbb{R}^n$, then $z = 0$ is globally exponential stable.

Proof. From (3.7) and (3.8), we have:

$$ {}^{\mathcal{CF}}_{0}\mathfrak{D}^{\zeta}_{v}\mathcal{V}(v,z(v)) \leqslant -\frac{\zeta_3}{\zeta_2}\mathcal{V}(v,z(v)), $$

hence, there exists a nonnegative function $K(v)$ such that

$$ {}^{CF}_{0}\mathfrak{D}^{\zeta}_{v}\mathcal{V}(v,z(v)) + K(v) = -\zeta_3\zeta_2^{-1}\mathcal{V}(v,z(v)). \tag{3.9}$$

Using Laplace transform operator on the both sides of (3.9), we obtain

$$\frac{\zeta M(\zeta)}{\zeta+s(1-\zeta)}(sL\{\mathcal{V}(v,z(v))\} - \mathcal{V}(0,z(0)) + L\{K(v)\} = -\frac{\zeta_3}{\zeta_2}L\{\mathcal{V}(v,z(v))\}$$

$$\overset{L\{\mathcal{V}(v,z(v))\}=\mathcal{V}(s)}{\underset{L\{K(v)\}=K(s)}{\longrightarrow}} \left(\frac{\zeta s M(\zeta)}{\zeta+s(1-\zeta)} + \frac{\zeta_3}{\zeta_2}\right)\mathcal{V}(s) = \frac{\zeta M(\zeta)}{\zeta+s(1-\zeta)}\mathcal{V}(0) - K(s)$$

Set

$$N := N(s,\zeta) = \frac{\zeta s M(\zeta)}{\zeta+s(1-\zeta)} + \frac{\zeta_3}{\zeta_2},$$

then we obtain

$$\mathcal{V}(s) = \frac{1}{N}\left[\frac{\zeta M(\zeta)}{\zeta+s(1-\zeta)}\mathcal{V}(0) - K(s)\right].$$

If $z(0)=0$, then $\mathcal{V}(0)=\mathcal{V}(0,z(0))=0$, and the solution of (3.6) is $z=0$. If $z(0)\neq 0, \mathcal{V}(0)>0$, since $\mathcal{V}(v,z)$ is locally Lipschitz w.r.t. z, it follows from the fractional existence and uniqueness theorem and the inverse Laplace transform, the unique solution of (3.9) is as follows:

$$\mathcal{V}(v,z(v)) = \zeta M(\zeta)V(0)L^{-1}\left\{\frac{1}{N.(\zeta+s(1-\zeta))}\right\} - L^{-1}\left\{\frac{K(s)}{N}\right\}.$$

In this step, we aim to prove that $L^{-1}\{\frac{K(s)}{N}\}$ is non-negative, then by employing this property, we will complete the proof of the lemma. Therefore, we have:

$$\begin{aligned} L^{-1}\left\{\frac{K(s)}{N}\right\} &= L^{-1}\left\{\frac{K(s)}{\frac{\zeta s M(\zeta)}{\zeta+s(1-\zeta)}+\frac{\zeta_3}{\zeta_2}}\right\} = L^{-1}\left\{\frac{K(s)}{\frac{\zeta_2\zeta s M(\zeta)+\zeta_3(\zeta+s(1-\zeta))}{\zeta_2(\zeta+s(1-\zeta))}}\right\} \\ &= L^{-1}\left\{\frac{\zeta_2(\zeta+s(1-\zeta))K(s)}{\zeta_2\zeta s M(\zeta)+\zeta_2\zeta+\zeta_3 s(1-\zeta)}\right\} \\ &= L^{-1}\left\{\frac{\zeta_2\zeta K(s)+\zeta_2 s(1-\zeta)K(s)}{s(\zeta_2\zeta M(\zeta)+\zeta_3-\zeta\zeta_3)+\zeta\zeta_3}\right\} \\ &= \zeta_2\zeta L^{-1}\left\{\frac{K(s)}{sH(\zeta)+d(\zeta)}\right\} \\ &\quad + \zeta_2(1-\zeta)L^{-1}\left\{\frac{sK(s)}{sH(\zeta)+d(\zeta)}\right\}, \end{aligned}$$

where $H(\zeta)=\zeta_2\zeta M(\zeta)+\zeta_3-\zeta\zeta_3$ and $d(\zeta)=\zeta\zeta_3$. After a number of simple calculations, we obtain:

$$L^{-1}\left\{\frac{K(s)}{N(s,\zeta)}\right\} = \frac{\zeta_2\alpha}{H(\zeta)}L^{-1}\left\{\frac{K(s)}{s+\frac{d(\zeta)}{H(\zeta)}}\right\} + \frac{\zeta_2(1-\zeta)}{H(\zeta)}L^{-1}\left\{\frac{sK(s)}{s+\frac{d(\zeta)}{H(\zeta)}}\right\} \tag{310}$$

Using the property $F(s-a)=L\{e^{at}f(t)\}$, we obtain $L^{-1}\{\frac{K(s)}{s+A(\zeta)}\}$, where $A(\zeta)=\frac{d(\zeta)}{H(\zeta)}$ as follows:

$$L^{-1}\left\{\frac{K(s)}{s+A(\zeta)}\right\} \overset{s\to s-A(\zeta)}{=} L^{-1}\left\{\frac{K(s-A(\zeta))}{s}\right\} = L^{-1}\left\{\frac{L\{e^{A(\zeta)}K(t)\}}{s}\right\}$$

$$= e^{A(\zeta)}L^{-1}\left\{\frac{L\{K(v)\}}{s}\right\}$$

Using the property $L^{-1}\{sG(s)\} = g'(v)+g(0)$, we have:

$$L^{-1}\left\{\frac{sK(s)}{s+A(\zeta)}\right\} = p'(v)+p(0),$$

where $p(v) = L^{-1}\{\frac{K(s)}{s+A(\zeta)}\} = e^{A(\zeta)}\int_0^v K(z)dz \geqslant 0$ and $p(0)=0$.

Thus, $p'(v)=e^{A(\zeta)}K(v)$ and finally we have:

$$L^{-1}\left\{\frac{sK(s)}{s+A(\zeta)}\right\} = e^{A(\zeta)}K(v) \geqslant 0$$

As it is obvious, all the terms of the Eq. (310) are non-negative. So, we can conclude that $L^{-1}\{\frac{K(s)}{N(s,\zeta)}\}$ is non-negative.

Since $L^{-1}\{\frac{K(s)}{N}\}$ is non-negative, we have:

$$\mathcal{V}(v,z(v)) \leqslant \zeta M(\zeta)\mathcal{V}(0)L^{-1}\left\{\frac{1}{N.(\zeta+s(1-\zeta))}\right\}.$$

Hence

$$\mathcal{V}(v,z(v)) \leqslant \zeta M(\zeta)\mathcal{V}(0)\frac{\zeta_2 e^{\frac{-t(\zeta\zeta_3+M(\zeta)\zeta_2)}{\zeta_3-\zeta\zeta_3}}}{\zeta_3-\zeta\zeta_3}. \tag{3.11}$$

Using Eq. (3.11) and Eq. (3.7), we obtain:

$$||z(v)|| \leqslant \left[\frac{\mathcal{V}(0)}{\beta_1}\exp[-v(\beta_2)]\right]^{\frac{1}{c}}$$

where

$$\beta_1 = \frac{c(\zeta_3-\zeta\zeta_3)}{\zeta M(\zeta)\zeta_2} \quad \& \quad \beta_2 = \frac{\zeta\zeta_3+M(\zeta)\zeta_2}{\zeta_3-\zeta\zeta_3}$$

which imply the exponential stability of system (3.6). □

Now we state a foremost inequality for CF-derivative of a composite function.

Theorem 3.1. Let $\mathcal{V}(v,z(v)):\Omega\to\mathbb{R}$ and $z(v):[v_0,\infty)\to\Omega$ are two continuous and differentiable functions, where $\Omega\subset\mathbb{R}^n$ is a set. Suppose that $\mathcal{V}(v,z(v))$ is convex over Ω. Therefore, for any time instant $v\geqslant v_0$,

$$ {}^{CF}_{v_0}\mathfrak{D}^{\zeta}_{v}\mathcal{V}(z(v)) \leqslant \left(\frac{\partial\mathcal{V}}{\partial z}\right)^T {}^{CF}_{v_0}\mathfrak{D}^{\zeta}_{v}z(v), \quad \forall\zeta\in(0,1). \tag{3.12}$$

Proof. The inequality (3.12) is equivalent to:

$$ {}^{CF}_{v_0}\mathfrak{D}^{\zeta}_{v}\mathcal{V}(z(v)) - \left(\frac{\partial\mathcal{V}}{\partial z}\right)^T {}^{CF}_{v_0}\mathfrak{D}^{\zeta}_{v}z(v) \leqslant 0. \tag{3.13}$$

Using Definition 2.1, the inequality (3.13) can be written by:

$$\frac{\zeta M(\zeta)}{1-\zeta}\int_{t_0}^{t}\mathcal{V}'(z(s)).\exp[-\tfrac{\zeta}{1-\zeta}(v-s)]ds$$
$$-\left(\tfrac{\partial\mathcal{V}}{\partial z}\right)^T\frac{\zeta M(\zeta)}{1-\zeta}\int_{v_0}^{v}z'(s)\exp[-\tfrac{\zeta}{1-\zeta}(t-s)]ds \leqslant 0$$

Then

$$\frac{\zeta M(\zeta)}{1-\zeta}\int_{v_0}^{v}\left(\frac{\partial\boldsymbol{\mathcal{V}}(z(s))}{\partial z}-\frac{\partial\boldsymbol{\mathcal{V}}(z(v))}{\partial v}\right).z'(s).\exp[-\frac{\zeta}{1-\zeta}(v-s)]ds\leqslant 0 \tag{3.14}$$

Set

$$\Phi(s,v)=\boldsymbol{\mathcal{V}}(z(s))-\boldsymbol{\mathcal{V}}(z(v))-\left(\frac{\partial\boldsymbol{\mathcal{V}}}{\partial z}\right)^{T}(z(s)-z(v)).$$

Then Eq. (3.14) can be written as follows:

$$\frac{\zeta M(\zeta)}{1-\zeta}\int_{v_0}^{v}\frac{d}{ds}\Phi(s,v).\exp[-\frac{\zeta}{1-\zeta}(v-s)]ds$$

$$=\frac{\zeta M(\zeta)}{1-\zeta}\int_{v_0}^{v}d[\Phi(s,v)].\exp[-\frac{\zeta}{1-\zeta}(t-s)]ds\leqslant 0$$

Using the partial integral formula, we have:

$$\frac{\zeta M(\zeta)}{1-\zeta}\int_{v_0}^{v}d[\Phi(s,v)].\exp[-\frac{\zeta}{1-\zeta}(v-s)]ds= \tag{3.15}$$
$$\frac{\zeta M(\zeta)}{1-\zeta}\Bigg[-\exp[-\frac{\zeta}{1-\zeta}(v-v_0)]\Phi(v_0,v)$$
$$-\frac{\zeta}{1-\alpha}\int_{v_0}^{v}\Phi(s,v).\exp[-\frac{\zeta}{1-\zeta}(v-s)]ds\Bigg].$$

Since $\boldsymbol{\mathcal{V}}(v,z(v))$ is convex, we have:

$\Phi(s,v)\geqslant 0,$

thus, the last two terms in Eq. (3.15) is bounded by:

$$\frac{\zeta M(\zeta)}{1-\zeta}\Bigg[-\exp[-\frac{\zeta}{1-\zeta}(v-v_0)]\Phi(0,v)$$
$$-\frac{\zeta}{1-\zeta}\int_{v_0}^{v}\Phi(s,v).\exp[-\frac{\zeta}{1-\zeta}(v-s)]ds\Bigg]\leqslant 0$$

□

Now using Theorem 3.1, we prove a new result regarding the Lyapunov-based stability for fractional-order system with CF-derivative.

Theorem 3.2. Assume that $z=0$ is an equilibrium point for noninteger-order system (3.6). Also, suppose that there exists a convex Lyapunov function $\boldsymbol{\mathcal{V}}(v,z(v))$ and locally Lipschitz w.r.t. z such that

$$\gamma_1\|z\|^{c}\leqslant\boldsymbol{\mathcal{V}}(v,z(v))\leqslant\gamma_2\|z\|^{c}$$

$$\left(\frac{\partial\boldsymbol{\mathcal{V}}}{\partial z}\right)^{T}g(z,v)\leqslant-\gamma_3\|z\|^{cd}$$

where $\gamma_1,\gamma_2,\gamma_3,c$ and d are arbitrary positive constants. Then, $z=0$ is globally exponential stable.

Proof. Using Theorem 3.1, we obtain

$${}_{0}^{\mathcal{CF}}\mathfrak{D}_{v}^{\zeta}\boldsymbol{\mathcal{V}}(v,z(v))\leqslant\left(\frac{\partial\boldsymbol{\mathcal{V}}}{\partial z}\right)^{T}{}_{0}^{\mathcal{CF}}\mathfrak{D}_{v}^{\zeta}z(v)$$

$$=\left(\frac{\partial\boldsymbol{\mathcal{V}}}{\partial z}\right)^{T}g(z,v)$$

$$\leqslant-\gamma_3\|z\|^{cd}$$

Using Lemma 3.1, $z=0$ is globally exponential stable.

3.1. Class-k functions

Definition 3.1. A continuous function $\gamma:[0,v)\to[0,+\infty)$ is said to belong to class-k, if it is strictly increasing and $\gamma(0)=0$

Lemma 3.2. (Fractional comparison principle). Assume that $z(0)=u(0)$ and ${}^{\mathcal{CF}}\mathfrak{D}^{\zeta}z(v)\geqslant{}^{\mathcal{CF}}\mathfrak{D}^{\zeta}u(v)$, where $\zeta\in(0,1)$. Then $z(v)\geqslant u(v)$.

Proof. In fact, a non-negative $\psi(v)$ exists such that

$${}^{\mathcal{CF}}\mathfrak{D}^{\zeta}z(v)=\psi(v)+{}^{\mathcal{CF}}\mathfrak{D}^{\zeta}u(v).$$

Using Laplace transform operator, we get:

$$\frac{sL\{z(v)\}-z(0)}{s+\zeta(1-s)}=L\{\psi(v)\}+\frac{sL\{u(v)\}-u(0)}{s+\zeta(1-s)}$$

Set $Z(s)=Lz(v),H(s)=L\psi(v),U(s)=Lu(v)$ and using the assumption $z(0)=u(0)$ we obtain

$$Z(s)=\frac{s+\zeta(1-s)}{s}\Psi(s)+u(s)$$

Then, using inversion of Laplace transform operator, L^{-1}, we have:

$$z(v)={}^{\mathcal{CF}}I^{\zeta}\psi(v)+u(v).$$

Hence, using the fact that $\psi(v)\geqslant 0$, we deduce that $z(v)\geqslant u(v)$. □

Theorem 3.3. Assume that $z=0$ is an equilibrium point for the non-autonomous fractional order system (3.6). Suppose that a Lyapunov function $\boldsymbol{\mathcal{V}}(v,z(v))$ exists and class-k functions $\gamma_i(i=1,2,3)$ such that

$$\gamma_1(\|z\|)\leqslant\boldsymbol{\mathcal{V}}(v,z(v))\leqslant\gamma_2(\|z\|), \tag{3.16}$$

$${}^{\mathcal{CF}}\mathfrak{D}^{\zeta}\boldsymbol{\mathcal{V}}(v,z(v))\leqslant-\gamma_3(\|z\|), \tag{3.17}$$

where $\zeta\in(0,1)$. Then, the equilibrium point of system (3.6) is asymptotically stable.

Proof. Using Eqs. (3.16) and (3.17), we have:

$${}^{\mathcal{CF}}\mathfrak{D}^{\zeta}\boldsymbol{\mathcal{V}}(v,z(v))\leqslant-\gamma_3(\gamma_2^{-1}(\boldsymbol{\mathcal{V}}(v,z(v))). \tag{3.18}$$

Then, using Lemma 3.2 and $\boldsymbol{\mathcal{V}}(v,z(v))\geqslant 0$, we have:

$\boldsymbol{\mathcal{V}}(t,z(t))\leqslant\boldsymbol{\mathcal{V}}(0,z(0)).$

Using the approach proposed in [40], we have:

Case I: Let assume that $v_1\geqslant 0$ exists which is satisfying $\boldsymbol{\mathcal{V}}(v_1,z(v_1))=0$. Using Eq. (3.16), we deduce that $z(v_1)=0$, and then $z(v)=0$ for $v\geqslant v_1$ (it comes from the fact that $z=0$ is equilibrium point of system (3.6)).

Case II: Let suppose that a positive constant ϵ exists such that $\boldsymbol{\mathcal{V}}(v,z)\geqslant\epsilon$, for $v\geqslant 0$. Then

$$0<\epsilon<\boldsymbol{\mathcal{V}}(v,z)\leqslant\boldsymbol{\mathcal{V}}(0,z(0)),\quad \text{for} v\geqslant 0. \tag{3.19}$$

Using Eqs. 3.16,3.17, we have:

$$^{\mathcal{CF}}\mathfrak{D}^{\zeta}\mathcal{V}(v,z(v)) \leqslant -\gamma_3(\gamma_2^{-1}(\mathcal{V}(v,z(v)))$$

$$\leqslant -\lambda\mathcal{V}(v,z(v))$$

where $\lambda = \frac{\gamma_3(\gamma_2^{-1}(\epsilon))}{\mathcal{V}(0,z(0))} > 0$. Hence

$$\mathcal{V}(v,z(v)) \leqslant \frac{\mathcal{V}(0,z(0))}{2-\zeta} e^{\frac{\zeta v}{2-\zeta}}$$

which contradicts with the hypothesis stats that $\mathcal{V}(v,z(v)) \geqslant \epsilon$.

Thus, under Case I and Case II, we get:

$$\lim_{v\to\infty}\mathcal{V}(v,z(v)) = 0$$

and finally we achieve:

$$\lim_{v\to\infty} z(v) = 0$$

□

4. Cases

Example 4.1. Let assume the next noninteger-order system [41]:

$$\begin{cases} {}^{\mathcal{CF}}_{v_0}\mathfrak{D}^{\zeta}_{v} z_1(v) = -z_1(v) - z_2^{\frac{1}{3}}(v), \\ {}^{\mathcal{CF}}_{v_0}\mathfrak{D}^{\zeta}_{v} z_2(v) = -z_1^{\frac{1}{3}}(v) - z_2(v). \end{cases} \tag{4.20}$$

By choosing a convex Lyapunov function

$$\mathcal{V}(v,z(v)) = z_1^{\frac{4}{3}}(v) + z_2^{\frac{4}{3}}(v),$$

where $z(v) = [z_1((v), z_2(v)]^T$, we have:

$$\begin{cases} \left(\frac{\partial\mathcal{V}}{\partial z}\right)^T g(z,v) \leqslant -\mathcal{V}(v,z(v)), \\ ||z(v)||^{\frac{4}{3}} \leqslant \mathcal{V}(v,z(v)) \leqslant \sqrt[3]{2}||z(v)||^{\frac{4}{3}}. \end{cases}$$

Then,

$$\left(\frac{\partial\mathcal{V}}{\partial z}\right)^T g(z,v) \leqslant -||z(v)||^{\frac{4}{3}}.$$

Therefore, we imply that the above system is globally exponential stable.

Example 4.2. Suppose the following CF-derivative equation of sliding surface:

$$^{\mathcal{CF}}\mathfrak{D}^{\zeta}(X(t)) = \eta\,\mathrm{sgn}(X(t)), \quad 0<\alpha<1, \eta>0. \tag{4.21}$$

Using the following Lyapunov function

$$\mathcal{V}(t) = \frac{1}{2}X^2(t),$$

we obtain

$$\dot{\mathcal{V}}(t) = X(t)\dot{X}(t).$$

Notice that

$$^{\mathcal{CF}}\mathfrak{D}^{\zeta}(X(t)) = \begin{cases} >0, & \textit{if } \dot{X}(t) > 0 \\ <0, & \textit{if } \dot{X}(t) < 0. \end{cases}$$

In order to discuss about the sign of $\dot{X}(t), \mathrm{sgn}(\dot{X}(t))$, we employ the following approach:

$$\begin{aligned} ^{\mathcal{CF}}\mathfrak{D}^{\zeta}(X(t)) = -\eta\,\mathrm{sgn}(X(t)) &\to {}^{\mathcal{CF}}I^{\zeta}({}^{\mathcal{CF}}\mathfrak{D}^{\zeta}(X(t))) \\ &= -\eta\,{}^{\mathcal{CF}}I^{\zeta}(\mathrm{sgn}(X(t)) \end{aligned}$$

$$\Rightarrow X(t) - X(0) = \begin{cases} -\eta\left(\frac{1-\zeta}{M(\zeta)} + \frac{\zeta t}{M(\zeta)}\right), & X(t) > 0 \\ \eta\left(\frac{1-\zeta}{M(\zeta)} + \frac{\zeta t}{M(\zeta)}\right), & X(t) < 0. \end{cases}$$

Therefore,

$$\dot{X}(t) = \begin{cases} -\eta\frac{\zeta}{M(\zeta)}, & \textit{If } \mathrm{sgn}(X(t)) > 0 \\ \eta\frac{\zeta}{M(\zeta)}, & \textit{If } \mathrm{sgn}(X(t)) < 0. \end{cases}$$

Indeed, it is easy to verify that $\eta\frac{\zeta}{M(\zeta)} > 0$, then,

$$\dot{X}(t) = \begin{cases} <0, & \textit{If } \mathrm{sgn}(X(t)) > 0 \\ >0, & \textit{If } \mathrm{sgn}(X(t)) < 0 \end{cases}$$

Hence, $\mathrm{sgn}(\dot{X}(t)) = -\mathrm{sgn}(X(t))$. Now, we have:

$$\begin{aligned} &\mathrm{sgn}(\dot{\mathcal{V}}(t)) = \mathrm{sgn}(X(t))\,\mathrm{sgn}(\dot{X}(t)) \\ &= -\mathrm{sgn}(X(t))\,\mathrm{sgn}(\dot{X}(t)) < 0. \end{aligned}$$

In fact, if $\dot{\mathcal{V}}(t)$, then $^{\mathcal{CF}}\mathfrak{D}^{\zeta}(\mathcal{V}(t)) < 0$. It is easy to verify that $\mathcal{V}(t) = \frac{1}{2}X^2(t) \geqslant 0$. Hence, using Lemma 3.1, $X = 0$ is globally exponential stable.

Remark 4.1. Example 4.2 was has been solved in [43] using Caputo-type differentiability.

5. Conclusion

In this report, we analysed the stability conditions of fractional nonlinear systems with noninteger-order CF-derivative. We argued the fractional nonautonomous systems and investigated Lipschitz condition for fractional-order systems. We introduced the definition of exponential and Lyapunov stability for the non-singular kernel derivative, which enhanced the information of system theory and the FC. We established the fractional comparison principle for aforesaid derivative based on the class-k functions. Two cases were experienced to support the validity of the proposed methodology.

Our future investigations will involve the stability analysis of fractional nonautonomous systems with the Hilfer fractional derivative and also we will expand the achieved results for a variety of fractional differential equations under interval arithmetic.

Declaration of Competing Interest

None.

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