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STABILITY AND HOPF BIFURCATION ANALYSIS OF A DISTRIBUTED TIME DELAY ENERGY MODEL FOR SUSTAINABLE ECONOMIC GROWTH

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ABSTRACT. This paper examines the consequences of including distributed delays in an energy model. The stability behaviour of the resulting equilibrium for our dynamic system is analysed, including models with Dirac, weak and strong kernels. Applying the Hopf bifurcation theorem we determine conditions under which limit cycle motion is born in such models. The results indicate that distributed delays have an ambivalent impact on the dynamical behaviour of systems, either stabilizing or destabilizing them.

1. Introduction

Domar (1946) and Solow (1956) introduced the systematic study of the growth process, where the accumulation of physical capital is seen as a key growth driver. This research unveiled key structural characteristics that affect the long-term determination of labor productivity: savings, population growth, technological change. In this regard, different augmented versions of the Solow model were developed over the decades that include among others the contribution of Mankiw *et al.* (1992) employing a physical and an intangible capital (human capital), and the Solow-Jorgenson-Griliches residual model (Jorgenson and Griliches 1967), framework used to address issues concerning intangibles such as the contribution of intangible capital to output growth and how does the inclusion of intangibles affect the allocation of output growth between capital formation and multifactor productivity growth (Corrado *et al.* 2009).

On the other hand, the basic neoclassical growth concept abstracts from a potentially important feature of the growth process: purchasing a piece of machinery at a specific time and place makes little sense unless the equipment can be supplied with energy and put to use. Indeed, it is of primary importance for the economy to be able to distribute electricity across the economy.

Natural science experiments by Banavar *et al.* (1999, 2002) and West *et al.* (1997, 1999) developed the energy distribution network as a living organism to validate the empirical results of biology recognized as. Kleiber's Law. Dalgaard and Strulik (2011) introduced this theory and were the first to integrate something as complicated as the electricity distribution network into a macroeconomic framework. They claimed that the application of biological organism-related concepts on man-made energy networks is a true strategy for the following three reasons: the cardiovascular system functions in the same manner as the electricity network, facilitating the transfer of nutrients and power around the network; man-made and biological networks have different added properties due to the development process over time, as processes refine their natural selection in the case of a biological network as well as constant rework and updating of human-made networks; physicists use empirical methods from biological organisms to look for universal laws of scaling that affect human society. The relentless pressure to move towards an optimum distribution system allows for certain characteristics to be shared between biological and man-made networks. If the consumption per capita of electricity is the equivalent of metabolism and capital per capita is equal to body mass, it is assumed that the relationship between capital and electricity is concave and log linear. Dalgaard and Strulik (2011) demonstrated that energy is accessible at geographically dispersed locations through an economy's self-organization. Further, they investigated the relationship between capital per capita and energy per capita from the energy demand viewpoint and claimed that energy demand is driven by the need for capital management, maintenance and generation.

Bianca *et al.* (2013) modified the model by Dalgaard and Strulik (2011) with the assumption that the energy conservation formula would be influenced by a time delay, thereby characterizing the dynamics of the system by the following delay differential equation,

$$\dot{k}(t) = \frac{\varepsilon}{v}\,[k(t-\tau)]^a - \frac{\mu}{v}k(t-\tau), \tag{1}$$

where k denotes per capita capital, μ and v are the energy required to operate and maintain the generic capital good, and the energy costs to create a new capital good, respectively, $a \in (0,1)$ and $\varepsilon > 0$ are real constants, and $\tau \geq 0$ represents a time delay. For $\tau = 0$, Eq. (1) reduces to a law of motion for capital which is structurally identical to that implied by the Solow model (Solow 1956). Since it shares its technical properties, there exists a unique stable steady-state $k_{*,}$ where $k_*^{a-1} = \mu/\varepsilon$, to which the economy adjusts. For $\tau > 0$, by choosing time delay as a bifurcation parameter, Bianca *et al.* (2013) proved that the model loses stability and a Hopf bifurcation occurs when time delay passes through critical values. It is known that time delays in economic situations can be modeled in two different ways: discrete time delays, ideal when a fixed time delay for the agents involved is institutionally or socially defined, and continuously distributed time delays, appropriate when the delay is uncertain or varying periods of delays are spread across the agents.

In this paper, following Guerrini *et al.* (2019), we generalize the delay differential equation model with a discrete delay, namely (1), adopting continuously distributed time delays. Accordingly, the model may be written as follows

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t} g(t-r,S,m)k(r)dr\right]^{a} - \frac{\mu}{v}\int_{-\infty}^{t} g(t-r,T,n)k(r)dr. \tag{2}$$

The function g in (2) is a non-negative bounded function defined on $[0,+\infty)$ which reflects the influence of the past states on the current dynamics and it is called the delay kernel. Here, S,T are positive parameters associated with the average length of the continuous delay and $m,n \in \{0,1\}$ determine the shape of the weighting function. In line with Cushing (1977), we consider the following types of gamma distribution for g,

$$g(t-r,\zeta,0) = \left(\frac{1}{\zeta}\right)e^{-\frac{1}{\zeta}(t-r)} \quad \text{and} \quad g(t-r,\zeta,1) = \left(\frac{2}{\zeta}\right)^{2}(t-r)e^{-\frac{2}{\zeta}(t-r)},$$

where $\zeta = S,T$, which are named *weak delay kernel* and *strong delay kernel*, respectively. In the former case, weights are exponentially declining with the most weight being given to the most recent output; in the latter one, zero weight is assigned to the most recent output, rising to maximum weight at a point ζ time units in the past and declining exponentially to zero thereafter. Notice that as $\zeta \to 0$, the function g tends to a Dirac delta function, i.e. $\delta(t-r)$, so that one recovers the discrete delay case (1) (with $\tau = T$). Therefore, Eq. (2), that we are interested in, is more general than Eq. (1). Since time delay does not change the equilibria of the equation, Eq. (2) has exactly the same equilibrium point of the standard Solow model ($\tau = 0$) since time delay does not change the equilibria of the equation. An analysis of the model (2), using a combination of the previous expressions for g as well Dirac kernel, will be done via the so-called linear chain trick technique (MacDonald 1978), which transforms the integrodifferential system (2) into an equivalent system of ordinary or delay differential equations. In this context, the principal role of delays is in destabilizing an otherwise stable economy and, depending upon a combination of two delays, the delay can also stabilise the economy. Our results will stress the importance of the theoretical modelling framework used as a device that may dramatically change the findings of the model in (1). In this context, the principal role of delay is in destabilizing an otherwise stable economy and the delay can, depending on the combination of 2 delays, also stabilise the economy. For future research, we propose to extend our model in a computational and experimental way, and generalize it to include the intangible asset of human capital.

2. Weak kernels

Let, $m = 0$ and $n = 0$. Eq. (2) rewrites as

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)dr\right]^{a} - \frac{\mu}{v}\int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)dr. \tag{3}$$

For convenience, we define the new variables $x(t)$ and $y(t)$ by

$$x(t) = \int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)dr, \qquad y(t) = \int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)dr.$$

Applying the linear chain trick technique, Eq. (3) can be transformed into the following third-dimensional system of ordinary differential equations

$$\begin{cases} \dot{k}(t) &= \frac{\varepsilon}{v}[x(t)]^{a} - \frac{\mu}{v}y(t), \\ \dot{x}(t) &= \frac{1}{S}[k(t) - x(t)], \\ \dot{y}(t) &= \frac{1}{T}[k(t) - y(t)]. \end{cases} \tag{4}$$

The local stability of the unique positive equilibrium point (k_*, x_*, y_*) of system (4), where $x_* = y_* = k_*$ and k_* is the steady-state of (1), is governed by the roots of the corresponding characteristic equation for system (4). Linearizing this system at its equilibrium point we obtain the characteristic equation

$$\lambda^3 + b_1(S,T)\lambda^2 + b_2(S,T)\lambda + b_3(S,T) = 0, \tag{5}$$

where

$$b_1(S,T) = \frac{S+T}{ST} > 0, \qquad b_2(S,T) = \frac{v + \mu(S - aT)}{ST}, \qquad b_3(S,T) = \frac{(1-a)\mu}{vST} > 0.$$

Lemma 1. *The equilibrium point k_* of (3) is locally asymptotically stable for $0 \leq T < T_*$ and unstable for $T > T_*$, where*

$$T_* = \frac{v + \sqrt{v^2 + 4a\mu S(v + \mu S)}}{2a\mu}.$$

Proof. By the Routh-Hurwitz criteria, the equilibrium point is locally asymptotically stable if and only if $b_1(S,T) > 0$, $b_3(S,T) > 0$ and $b_1(S,T)b_2(S,T) > b_3(S,T)$. Thus, the stability condition is confirmed if $b_1(S,T)b_2(S,T) > b_3(S,T)$. A direct calculation yields $a\mu T^2 - vT - S(v + \mu S) < 0$. The statement follows solving this inequality. ∎

We now return to the characteristic equation (5) and show the possibility of the birth of a limit cycle at $T = T_*$ by applying the Hopf bifurcation theorem. According to this theorem, we can establish the existence of a cyclic solution if the cubic characteristic equation has a pair of pure imaginary roots and the real parts of these roots change signs with a bifurcation parameter. At the critical value T_* one has $b_1(S,T_*)b_2(S,T_*) = b_3(S,T_*)$, and the characteristic equation can be rewritten as

$$[\lambda + b_1(S,T_*)]\left[\lambda^2 + b_2(S,T_*)\right] = 0,$$

which has roots

$$\lambda_1 = -b_1(S,T_*) < 0, \qquad \lambda_{2,3} = \pm i\omega_*, \text{ with } \omega_* = \sqrt{b_2(S,T_*)}.$$

Recalling that the equilibrium is locally asymptotically stable in absence of delays, if the transversality condition

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} > 0$$

holds, then a Hopf bifurcation occurs at the equilibrium point when T passes through the critical value T_*. Differentiating Eq. (5) with respect to T, and using (5), we have

$$\frac{d\lambda}{dT} = \frac{-b_1'(S,T)\lambda^2 - b_2'(S,T)\lambda}{3\lambda^2 + 2b_1(S,T)\lambda + b_2(S,T)},$$

where

$$b_1'(S,T) = -\frac{v+\mu S}{ST^2}, \qquad b_2'(S,T) = -\frac{(1-a)\mu}{vST^2}.$$

Since $\omega_*^2 = b_2(S,T_*)$ and $b_1(S,T_*)b_2(S,T_*) = b_3(S,T_*)$, after some calculations, we get

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} = -\frac{b_1'(S,T_*)b_2(S,T_*) + b_1(S,T_*)b_2'(S,T_*)}{2\left[b_2(S,T_*) + b_1^2(S,T_*)\right]}.$$

Since the numerator in the above expression is equal to $S[a\mu T_*^2 + S(v+\mu S)]/v > 0$, we conclude that the crossing direction of characteristic root through the imaginary axis is from right to left as T increases. Summarizing the above analysis, we have the following result.

Theorem 1. *Eq.* (3) *undergoes a Hopf bifurcation at its equilibrium point k_* when $T = T_*$.*

3. Weak and strong kernels

Let $m = 0$ and $n = 1$. Eq. (2) becomes

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)dr\right]^a - \frac{\mu}{v}\int_{-\infty}^{t}\left(\frac{2}{T}\right)^2(t-r)e^{-\frac{2}{T}(t-r)}k(r)dr. \qquad (6)$$

Defining the new variables $x(t), y(t)$ and $z(t)$ by

$$x(t) = \int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)dr, \qquad y(t) = \int_{-\infty}^{t}\left(\frac{2}{T}\right)^2(t-r)e^{-\frac{2}{T}(t-r)}k(r)dr,$$

and

$$z(t) = \int_{-\infty}^{t} \left(\frac{2}{T}\right) e^{-\frac{2}{T}(t-r)} k(r)dr,$$

Eq. (6) is turned into the following fourth-dimensional system of ordinary differential equations

$$\begin{cases} \dot{k}(t) &=& \dfrac{\varepsilon}{v}\,[x(t)]^{a} - \dfrac{\mu}{v} y(t), \\ \dot{x}(t) &=& \dfrac{1}{S}\,[k(t) - x(t)], \\ \dot{y}(t) &=& \dfrac{2}{T}\,[z(t) - y(t)], \\ \dot{z}(t) &=& \dfrac{2}{T}\,[k(t) - z(t)]. \end{cases} \tag{7}$$

The characteristic equation for system (7) at the equilibrium point (k_*, x_*, y_*, z_*), where $x_* = y_* = z_* = k_*$, takes the form

$$\lambda^4 + c_1(S,T)\lambda^3 + c_2(S,T)\lambda^2 + c_3(S,T)\lambda + c_4(S,T) = 0, \tag{8}$$

where

$$c_1(S,T) = \frac{4S+T}{ST} > 0, \qquad c_2(S,T) = \frac{4v\,(S+T) - a\mu T^2}{vST^2},$$

$$c_3(S,T) = \frac{4v + 4\mu\,(S - aT)}{vST^2}, \qquad c_4(S,T) = \frac{4\,(1-a)\,\mu}{vST^2} > 0.$$

Now, it is necessary to investigate the distribution of roots of Eq. (8) in order to determine the stability of the equilibrium.

Lemma 2. *The equilibrium point k_* of* (6) *is locally asymptotically stable for $a < (v + S\mu)/(\mu T)$ and*

$$\varphi(T) = c_1(S,T)c_2(S,T)c_3(S,T) - c_3^2(S,T) - c_1^2(S,T)c_4(S,T) > 0. \tag{9}$$

Proof. Using the Routh-Hurwitz criteria, all roots of the polynomial in (8) are negative or have negative real parts if and only if the following conditions hold: $c_1(S,T) > 0$, $c_3(S,T) > 0$, $c_4(S,T) > 0$ and $c_1(S,T)c_2(S,T)c_3(S,T) > c_3^2(S,T) + c_1^2(S,T)c_4(S,T)$, yielding the statement. ■

Remark 1. *Condition* (9) *is equivalent to*

$$\left(a^2\mu^2\right)T^4 - \left(v\mu + 4av\mu + Sa\mu^2\right)T^3 + 4\left(S^2a\mu^2 - Sv\mu - 2Sav\mu + v^2\right)T^2$$
$$+ 4S\left(-S^2\mu^2 - Sv\mu + 4v^2\right)T + 16vS^2\left(\mu S + v\right) > 0.$$

Next, we select T as the bifurcation parameter and show possibility of the birth of limit cycles when $T = T_*$. Let $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be the roots of the characteristic equation (8). Then, we have

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = -c_1(S,T), \qquad \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4 = c_2(S,T), \tag{10}$$

$$\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_4 = -c_3(S,T), \qquad \lambda_1\lambda_2\lambda_3\lambda_4 = c_4(S,T). \tag{11}$$

If there is $T = T_*$ such that $\varphi(T_*) = 0$, then by the Routh–Hurwitz criterion at least one root, say λ_1, has real part equal to zero. From the second equation of (11) it follows that $Im\lambda_1 = \omega_1 \neq 0$, so that there is another root, say λ_2, such that $\lambda_2 = \bar{\lambda}_1$. Since $\varphi(T)$ is a continuous function of its roots, λ_1 and λ_2 are complex conjugates in an open interval including T_*. A a result, the equations.in(10) and (11) have the following form at $T = T_*$,

$$\lambda_3+\lambda_4 = -c_1(S,T_*), \qquad \omega_1^2+\lambda_3\lambda_4 = c_2(S,T_*), \tag{12}$$

$$\omega_1^2(\lambda_3+\lambda_4) = -c_3(S,T_*), \qquad \omega_1^2\lambda_3\lambda_4 = c_4(S,T_*). \tag{13}$$

If λ_3 and λ_4 are complex conjugates, from the first equation of (12) we derive that $2Re\lambda_3 = -c_1(S,T_*) < 0$. If λ_3 and λ_4 are real, from the first and the second equation of (12) and (13), respectively, we obtain that $\lambda_3 < 0$ and $\lambda_4 < 0$. According to the Hopf bifurcation Theorem, it remains to verify the transversality condition. Finding the derivative on both sides of (8).with respect to T, we have

$$\frac{d\lambda}{dT} = -\frac{c_1'(S,T)\lambda^3+c_2'(S,T)\lambda^2+c_3'(S,T)\lambda+c_4'(S,T)}{4\lambda^3+3c_1(S,T)\lambda^2+2c_2(S,T)\lambda+c_3(S,T)}, \tag{14}$$

where

$$c_1'(S,T) = -\frac{4}{T^2}, \qquad c_2'(S,T) = -\frac{4(2S+T)}{ST^3},$$

$$c_3'(S,T) = \frac{4a\mu T-8(v+\mu S)}{vST^3}, \qquad c_4'(S,T) = -\frac{8(1-a)\mu}{vST^3}.$$

Since $\omega_* = c_3(S,T_*)/c_1(S,T_*)$, it follows from (14) that

$$\left(\frac{d\lambda}{dT}\right)_{T=T_*} =$$

$$\frac{[c_1'(S,T_*)c_3(S,T_*)-c_1(S,T_*)c_3'(S,T_*)]\,i\omega_*+c_2'(S,T_*)c_3(S,T_*)-c_1(S,T_*)c_4'(S,T_*)}{2\left\{[c_1(S,T_*)c_2(S,T_*)-2c_3(S,T_*)]\,i\omega_*-c_1(S,T_*)c_3(S,T_*)\right\}} \tag{15}$$

Multiplying both numerator and denominator of (15) by the conjugate of the denominator, and recalling that $\varphi(T_*) = 0$, i.e. $c_3^{*2} = c_1^* c_2^* c_3^* - c_1^{*2} c_4^*$, after a long calculation we get

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} = -\frac{c_1(S,T_*)\varphi'(T_*)}{2\left\{c_1^3(S,T_*)c_3(S,T_*) + [c_1(S,T_*)c_2(S,T_*) - 2c_3(S,T_*)]^2\right\}}.$$

As T increases, a positive (resp. negative) sign of (15) implies crossing of the imaginary axis from left to right (resp. from right to left). Thus, we have the following result.

Theorem 2. *Assume that $a < (v+S\mu)/(\mu T)$ and $\varphi(T) > 0$, where $\varphi(T)$ is defined as in* (9)*. If there exists $T = T_*$ such that $\varphi(T_*) = 0$ and $\varphi'(T_*) < 0$, then a Hopf bifurcation occurs at the equilibrium point k_* of* (6) *as T passes through T_*.*

4. Weak and Dirac kernels

Let $m = 0$ and $T \to 0$. Eq. (2) takes the form

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)dr\right]^a - \frac{\mu}{v}k(t-T). \tag{16}$$

Setting

$$x(t) = \int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)dr,$$

Eq. (16) takes the form of a second-dimensional system of delay differential equations

$$\begin{cases} \dot{k}(t) &= \dfrac{\varepsilon}{v}[x(t)]^a - \dfrac{\mu}{v}k(t-T), \\ \dot{x}(t) &= \dfrac{1}{S}[k(t) - x(t)]. \end{cases} \tag{17}$$

The associated characteristic equation of the linearization of (17) at the equilibrium point (k_*, x_*), where $x_* = k_*$, is

$$\lambda^2 + \frac{1}{S}\lambda - \frac{a\mu}{vS} + \left(\frac{\mu}{vS} + \frac{\mu}{v}\lambda\right)e^{-\lambda T} = 0. \tag{18}$$

Lemma 3. *Let $T = 0$. The equilibrium point k_* of* (16) *is locally asymptotically stable.*

Proof. In absence of delay, (18) reduces to

$$\lambda^2 + \left(\frac{1}{S} + \frac{\mu}{v}\right)\lambda + \frac{(1-a)\mu}{vS} = 0.$$

The conclusion is a straightforward matter being the coefficients both positive. ■

For the case $T > 0$, we determine parameter values for which (18) may have pure complex roots. We seek $\omega > 0$ such that $\lambda = i\omega$ satisfies (17). Substituting into (17), recalling that $e^{-i\omega T} = \cos(\omega T) - \sin(\omega T)$, and equating real and imaginary parts, we find that ω must simultaneously satisfy

$$\left\{ \begin{array}{lcl} \omega^2 + \dfrac{a\mu}{vS} & = & \dfrac{\mu}{vS}\cos(\omega T) + \dfrac{\mu}{v}\omega\sin(\omega T), \\ \dfrac{1}{S}\omega & = & \dfrac{\mu}{vS}\sin(\omega T) - \dfrac{\mu}{v}\omega\cos(\omega T). \end{array} \right. \tag{19}$$

Recalling that $\sin^2(\omega T) + \cos^2(\omega T) = 1$, squaring both sides of the equations in (19), adding and rearranging gives

$$\omega^4 - \left(\frac{\mu^2 S^2 - 2a\mu vS - v^2}{v^2 S^2}\right)\omega^2 - \frac{(1-a^2)\mu^2}{v^2 S^2} = 0.$$

This equation in ω^2 has a unique positive root, say ω_0, where

$$\omega_0^2 = \frac{\left(\mu^2 S^2 - 2a\mu vS - v^2\right)^2 + v^2 S^2 \sqrt{\left(\mu^2 S^2 - 2a\mu vS - v^2\right)^2 + 4(1-a^2)\mu^2 v^2 S^2}}{2v^4 S^4}. \tag{20}$$

Solving for $\cos(\omega T)$ and $\sin(\omega T)$ in (19) yields

$$\cos(\omega T) = \frac{a}{1 + S^2\omega^2} > 0, \qquad \sin(\omega T) = \frac{vS^2\omega^3 + (v + a\mu S)\omega}{\mu(1 + S^2\omega^2)} > 0. \tag{21}$$

Therefore, from (21) we see that

$$T_j = \frac{1}{\omega_0}\left[\cos^{-1}\left(\frac{a}{1 + S^2\omega_0^2}\right) + 2j\pi\right], \quad j = 0, 1, 2, ... \tag{22}$$

are the critical values of T for which the characteristic equation (18) has purely imaginary roots $\lambda = \pm i\omega_0$.

Let $\lambda(T) = u(T) + i\omega(T)$ be the root of (18) such that $u(T_j) = 0$ and $\omega(T_j) = \omega_0$. By differentiating (18) implicitly with respect to T, we get

$$\left[2vS\lambda + v + \mu S e^{-\lambda T} - \mu(1 + S\lambda) T e^{-\lambda T}\right]\frac{d\lambda}{dT} = \mu(1 + S\lambda)\lambda e^{-\lambda T}. \tag{23}$$

Hence, we have

$$\left(\frac{d\lambda}{dT}\right)^{-1} = \frac{(2S\lambda + 1) v e^{\lambda T} + \mu S}{\mu(1 + S\lambda)\lambda} - \frac{T}{\lambda}.$$

Then,

$$\begin{aligned} sign\left[\left.\frac{d(Re\lambda)}{dT}\right|_{T=T_j}\right] &= sign\left[Re\left.\left(\frac{d\lambda}{dT}\right)^{-1}\right|_{T=T_j}\right], \\ &= sign\left[\sqrt{\left(\mu^2 S^2 - 2a\mu vS - v^2\right)^2 + 4(1-a^2)\mu^2 v^2 S^2}\right] > 0. \end{aligned}$$

We summarise the foregoing discussion in the form of the following theorem.

Theorem 3. *Let T_0 be defined as in (20). The equilibrium point k_* of (16) is locally asymptotically stable when $0 < T \leq T_0$ and unstable when $T > T_0$. A Hopf bifurcation occurs at the equilibrium when $T = T_0$.*

5. Strong and weak kernels

Let $m = 1$ and $n = 0$. Eq. (2) is now governed by the following equation

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{2}{S}\right)^2 (t-r)e^{-\frac{2}{S}(t-r)}k(r)dr\right]^a - \frac{\mu}{v}\int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)dr. \tag{24}$$

Setting $x(t), y(t)$ and $z(t)$ by

$$x(t) = \int_{-\infty}^{t}\left(\frac{2}{S}\right)^2 (t-r)e^{-\frac{2}{S}(t-r)}k(r)dr, \quad y(t) = \int_{-\infty}^{t}\left(\frac{2}{S}\right)e^{-\frac{2}{S}(t-r)}k(r)dr$$

and

$$z(t) = \int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)dr.$$

Eq. (24) rewrites as a fourth-dimensional system of ordinary differential equations

$$\begin{cases} \dot{k}(t) &= \frac{\varepsilon}{v}[x(t)]^a - \frac{\mu}{v}z(t), \\ \dot{x}(t) &= \frac{2}{S}[y(t) - x(t)], \\ \dot{y}(t) &= \frac{2}{S}[k(t) - y(t)], \\ \dot{z}(t) &= \frac{1}{T}[k(t) - z(t)]. \end{cases} \tag{25}$$

In order to examine local dynamics of the above system in the neighborhood of the steady state (k_*, x_*, y_*, z_*), where $x_* = y_* = z_* = k_*$, we consider its linearized version and get the following characteristic equation

$$\lambda^4 + p_1(S,T)\lambda^3 + p_2(S,T)\lambda^2 + p_3(S,T)\lambda + p_4(S,T) = 0 \tag{26}$$

where

$$p_1(S,T) = \frac{S+4T}{ST} > 0, \qquad p_2(S,T) = \frac{S^2\mu + 4vS + 4v}{S^2Tv} > 0,$$

$$p_3(S,T) = \frac{4v + 4\mu(S - aT)}{vS^2T}, \qquad p_4(S,T) = \frac{4(1-a)\mu}{vS^2T} > 0.$$

Stability of (25) can be examined by finding the locations of the eigenvalues of Eq. (26). Comparing Eq. (26) with Eq. (8) reveals the similarities among them, and, consequently, analytical methodologies are similar. To avoid unnecessary repetition, the analysis of

(26) is simplified. Using first the Routh–Hurwitz criterion and then the Hopf bifurcation Theorem we arrive at the following results.

Theorem 4. *Let* $\psi(T) = p_1(S,T)p_2(S,T)p_3(S,T) - p_3^2(S,T) - p_1^2(S,T)p_4(S,T)$.

1) *The equilibrium point* k_* *of* (24) *is locally asymptotically stable if* $0 < a < (v+\mu S)/(\mu T)$ *and* $\psi(T) > 0$.
2) *If* $a < (v+S\mu)/(\mu T)$, $\psi(T) > 0$ *and there exists* $T = T_*$ *such that* $\psi(T_*) = 0$ *and* $\psi'(T_*) < 0$, *then the equilibrium point bifurcates to a limit cycle through a Hopf bifurcation at* T_*.

6. Dirac and weak kernels

Let $S \to 0$ and $n = 0$. Eq. (2) turns to be

$$\dot{k}(t) = \frac{\varepsilon}{v}[k(t-S)]^a - \frac{\mu}{v}\int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)dr. \tag{27}$$

Introducing the variable

$$x(t) = \int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{S}(t-r)}k(r)dr$$

allows (27) to be changed into the following two-dimensional system of delay differential equations

$$\begin{cases} \dot{k}(t) &= \dfrac{\varepsilon}{v}[k(t-S)]^a - \dfrac{\mu}{v}x(t), \\ \dot{x}(t) &= \dfrac{1}{T}[k(t)-x(t)]. \end{cases}$$

A straightforward calculation yields the following characteristic equation

$$\lambda^2 + \frac{1}{T}\lambda + \frac{\mu}{vT} - \frac{a\mu}{v}\left(\frac{1}{T}+\lambda\right)e^{-\lambda S} = 0. \tag{28}$$

Lemma 4. *Let* $S = 0$. *The equilibrium point* k_* *of* (27) *is locally asymptotically stable if* $0 \le T < v/(a\mu)$ *and unstable if* $T \ge v/(a\mu)$.

Proof. When $S = 0$, (28) reduces to

$$\lambda^2 + \left(\frac{1}{T} - \frac{a\mu}{v}\right)\lambda + \frac{(1-a)\mu}{vT} = 0.$$

Hence, we have the statement noticing that all the coefficients of this equation are positive when $T < v/(a\mu)$. ■

Now, let us take $S > 0$. We shall investigate the roots of the transcendental equation. (28) that lie in the left half of the complex plane. Suppose that $\lambda = i\omega,\ \omega > 0$, is a root of (28) for some S. Substituting this root into (28) and separating the real and imaginary parts implies

$$\begin{aligned} \omega^2 - \frac{\mu}{vT} &= -\frac{a\mu}{vT}\cos(\omega S) - \frac{a\mu}{v}\omega\sin(\omega S), \\ \frac{\omega}{T} &= -\frac{a\mu}{vT}\sin(\omega S) + \frac{a\mu}{v}\omega\cos(\omega S). \end{aligned} \tag{29}$$

Adding squares of these equations we obtain the following equation in ω^2,

$$\omega^4 + \left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{a^2\mu^2}{v^2}\right)\omega^2 + \frac{(1-a^2)\mu^2}{v^2T^2} = 0. \tag{30}$$

Proposition 1. *Let*

$$T_2 = \frac{v\left(-1+\sqrt{1-a^2}+\sqrt{2}\sqrt{1-\sqrt{1-a^2}}\right)}{a^2\mu}. \tag{31}$$

Then Eq. (28) has a pair of pure imaginary roots $\lambda = i\omega_\pm$, with $0 < \omega_- < \omega_+$, for $T_2 < T < v/(a\mu)$, where

$$\omega_\pm^2 = \frac{T^2a^2\mu^2 + 2Tv\mu - v^2 \pm \sqrt{a^4\mu^4T^4 + 4a^2v\mu^3T^3 + 2a^2v^2\mu^2T^2 - 4v^3\mu T + v^4}}{2T^2v^2}. \tag{32}$$

Proof. We start noticing that the constant term of (30) is positive. Therefore, solving for potential positive roots of (30) using the quadratic formula leads to the existence of two positive roots

$$\omega_\pm^2 = -\frac{1}{2}\left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{a^2\mu^2}{v^2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{a^2\mu^2}{v^2}\right)^2 - \frac{4(1-a^2)\mu^2}{v^2T^2}}$$

under the conditions

$$\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{a^2\mu^2}{v^2} < 0,$$

and

$$\Delta = \left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{a^2\mu^2}{v^2}\right)^2 - \frac{4(1-a^2)\mu^2}{v^2T^2} > 0.$$

The former condition gives

$$a^2\mu^2T^2 + 2\mu vT - v^2 > 0,$$

whose solution is given by

$$T > \frac{v\left(-1+\sqrt{1+a^2}\right)}{a^2\mu} = T_1.$$

The latter condition is instead verified when

$$\Delta > 0 \iff \frac{1}{T^2} - \frac{2\mu}{vT} - \frac{a^2\mu^2}{v^2} < -\frac{2\sqrt{1-a^2}\mu}{vT} \iff a^2\mu^2T^2 + 2\mu v\left(1-\sqrt{1-a^2}\right)T - v^2 > 0,$$

leading to $T > T_2$. This together with the fact that $T_1 < T_2 < v/(a\mu)$ completes the proof. ■

To find the corresponding critical values $S_j^{\pm}$ of S where the pure imaginary roots $i\omega_{\pm}$ exist, we solve (29) for $\sin(\omega S)$ and $\cos(\omega S)$, and get

$$\sin(\omega S) = \frac{\omega\left(-v + \mu T - vT^2\omega^2\right)}{a\mu\left(1+T^2\omega^2\right)}, \qquad \cos(\omega S) = \frac{1}{a\left(1+T^2\omega^2\right)} > 0.$$

By (32) one has

$$\sin(\omega S) = -\frac{\omega\left(v^2 + a^2\mu^2T^2 \pm \sqrt{a^4\mu^4T^4 + 4a^2v\mu^3T^3 + 2a^2v^2\mu^2T^2 - 4v^3\mu T + v^4}\right)}{4av\mu\left(1+T^2\omega^2\right)}$$

It is now immediate that $\sin(\omega_+ S) < 0$. On the other hand,

$$sign\left[\sin(\omega_- S)\right] = sign\left[-v^2 - a^2\mu^2T^2 + \sqrt{a^4\mu^4T^4 + 4a^2v\mu^3T^3 + 2a^2v^2\mu^2T^2 - 4v^3\mu T + v^4}\right].$$

A direct calculation shows this sign to be also negative as $T < v/(a\mu)$. Hence, $S_j^{\pm}$ $(j = 0,1,2,...)$ are defined by

$$S_j^{\pm} = \frac{1}{\omega_{\pm}}\left\{2\pi - \cos^{-1}\left[\frac{1}{a\left(1+T^2\omega_{\pm}\right)}\right] + 2j\pi\right\}.$$

Next, we check the validity of the transversality result. Differentiating (28) with respect to S, we have

$$\left\{2v\lambda + \frac{1}{T} - a\mu\left[1 - S\left(\frac{1}{T}+\lambda\right)\right]e^{-\lambda S}\right\}\frac{d\lambda}{dS} = -a\mu\lambda\left(\lambda + \frac{1}{T}\right)e^{-\lambda S}. \tag{33}$$

Then, using (28), it follows that

$$\left(\frac{d\lambda}{dS}\right)^{-1} = -\frac{a\mu - v\left(2\lambda + \frac{1}{T}\right)e^{\lambda S}}{a\mu\lambda\left(\lambda + \frac{1}{T}\right)} - \frac{S}{\lambda}.$$

Therefore,

$$\begin{aligned} sign\left[\left.\frac{d(Re\lambda)}{dS}\right|_{S=S_j^{\pm}}\right] &= sign\left[Re\left.\left(\frac{d\lambda}{dS}\right)^{-1}\right|_{S=S_j^{\pm}}\right], \\ &= sign\left[\pm\sqrt{a^4\mu^4T^4 + 4a^2v\mu^3T^3 + 2a^2v^2\mu^2T^2 - 4v^3\mu T + v^4}\right]. \end{aligned}$$

Hence the sign is positive for ω_+ and negative for ω_-. This implies that all the roots that cross the imaginary axis at $i\omega_+$ (resp. $i\omega_-$) cross from left to right (resp. from right to left) as S increases. It remains to prove that $\lambda = i\omega_\pm$ are simple roots for (28). Suppose $\lambda = i\omega_+$ (similarly for $i\omega_-$) is repeated, then from (33) one must have

$$-a\mu i\omega_+\left(i\omega_+ + \frac{1}{T}\right)\left[\cos\left(\omega_+ S_j^\pm\right) - i\sin\left(\omega_+ S_j^\pm\right)\right] = 0.$$

which clearly implies a contradiction. Thus, the conditions for a Hopf bifurcation are met and the Hopf bifurcation theorem holds. The above analysis is now summarized as follows.

Theorem 5. *Let T_2 be define as in* (31).

1) *Let $0 \le T \le T_2$. The equilibrium point k_* of* (27) *is locally asymptotically stable for all S.*
2) *Let $T_2 < T < v/(a\mu)$. Stability switches occur as the time delay S increases from zero to the positive infinity, with the occurrence of a Hopf bifurcation at each switch. If*
3) *Let $T \ge v/(a\mu)$. The equilibrium point k_* of* (27) *is unstable for all S.*

7. Concluding Remarks

We have considered different modelling approaches to study an energy model for sustainable economic growth when time delay is replaced by way of distributed delays. The system is modeled by gamma distributed delays, which includes the differential equations model with a discrete delay and the ordinary differential equations model as special cases. Employing the Routh-Hurwitz criterion and the results on distribution of the zeros of transcendental functions, we get a set of conditions to determine the stability of the equilibrium point and the existence of Hopf bifurcations. The choice of continuously distributed lag over a fixed time interval also yield the complex behaviour of emerging stability loss and gain which may repeat alternatively.

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