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Integral and differential approaches to Eringen's nonlocal elasticity models accounting for boundary effects with applications to beams in bending

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The Eringen's fully nonlocal elasticity model is known to lead to ill-posed boundary-value problems and to suffer some boundary effects arising from particle interactions impeded by the body's boundary surface. An enhanced model is derived from the original fully nonlocal one by the addition of a regularizing non-homogeneous local phase which accounts for boundary effects and which leads to a Fredholm integral equation of the second kind, hence to well-posed boundary-value problems, without paradoxes, nor other drawbacks. The enhanced integral model applied to a beam in bending proves to be equivalent to a sixth order differential equation with variable coefficients, with extra nonlocality boundary conditions here also derived. Both the integral approach and the differential one lead to a same unique solution of the small-scale beam problem. An efficient numerical algorithm is presented in which the sixth order differential equation with variable coefficients is reduced to one of the second order, which is addressed by a finite difference method. The proposed theory is applied to a set of engineering beam problems, for each of which the inherent size effects are reported and graphically illustrated. The influence of the length scale parameter upon the beam's response is highlighted by means of a function $\delta(\lambda)$ representing the normalized maximum deflection of the beam as a function of the length scale parameter. It is shown that the enhanced model always predicts *softening size effects* no matter the boundary and loading conditions, and that the related response function $\delta(\lambda)$ generally exhibits a *waved pattern with positive slopes first, then negative*, as the length scale parameter increases, with a limit asymptotic behavior like an atomic lattice model. A comparison with other theories is also presented together with possible future developments.

KEYWORDS
boundary effects, Euler–Bernoulli beam, nonlocal differential elasticity, nonlocal integral elasticity, size effects

1 | INTRODUCTION

The Eringen's nonlocal elasticity theory [1–4] has received notable consideration through the years for its conceptual and computational features that make it an attractive analytical tool for the analysis of size effects in small scale structures

[4–9]. In the last 15 years, after the advent of micro- and nano-technologies, the Eringen nonlocal theory (either in its integral, or differential, form) has been extensively applied to the study of beams, plates and shells simulating small scale specimens as sensors and actuators [10–21]. There is a huge literature in this field for which we make reference to the review papers by [22–26] and the references therein.

Peddieson and co-workers [27] first applied the differential nonlocal theory to beam structures and discovered some "paradoxes" in the obtained solutions (inability of the model to capture size effects as in the case of a cantilever beam under a point load; prediction of stiffening, or softening, size effects under different boundary and load conditions apparently without a precise physical rule). Also, an implausible increase of the vibration frequencies with increasing the length scale parameter of micro- and nano-beams was reported by [24, 28]. These discoveries led to extensive discussions on the Eringen nonlocal theory and to the proposal of alternative formulations as remedies to the undesirable anomalies. For instance, in [29–36] a two-phase local/nonlocal mixture model previously advanced by [1, 3] is used in place of the Eringen (fully) nonlocal one; in [37, 38] a strain-difference based nonlocal model is used, which grounds on a Helmholtz free energy and obeys the locality recovery condition; in [28, 39, 40] the Eringen fully nonlocal model is used, but the solution is searched out of the customary displacement continuity framework; in [30, 41–51] a nonlocal strain gradient model is employed, formed up by nonlocal and gradient constituents; in [52] a variational formulation of the nonlocal strain gradient theory for elastic materials of bi-Helmholtz type [53] is presented; in [33, 54] the solution is determined by the equivalent Helmholtz differential equation with the aid of nonlocality boundary conditions induced by integral equation theory [55, 56]; in [57–59] the integral equation theory is employed to address the Eringen nonlocal beam problem; in [60–66] the stress-driven nonlocal model is proposed and discussed, which is formally expressed like the Eringen fully nonlocal model, but the stress and strain state variables play interchanged roles; in [67] a review on the formulation of nonlocal elasticity problems is presented; in [68] a nonlocal model based on a modified kernel is employed.

The Eringen nonlocal integro-differential theory was discussed in a recent paper by the authors [69], with a particular concern to the inherent anomalies, which can be summarized as follows:

i. The Eringen (fully) nonlocal integral theory leads to a Fredholm integral equation of the first kind, which is known to lead to ill-posed boundary-value problems [55, 56].
ii. The Eringen nonlocal integral constitutive model does not comply with the locality recovery condition [38] according to which a nonlocal material has to behave as a classical local one whenever the source strain field is uniform, which means that, correspondingly, the Hooke law has to be recovered and the inherent Helmholtz free energy has to become independent of the length scale parameter.
iii. When applied to a one-dimensional problem like a beam problem, the differential form of the Eringen's nonlocal theory leads to a fourth order differential equation and to a unique solution of the beam problem, but this solution is not the solution of the corresponding Fredholm integral equation because the extra nonlocality boundary conditions cannot be satisfied. Generally the beam equilibrium equations impede that these extra boundary conditions may be satisfied, in which case the Fredholm integral equation (of the first kind) does not admit a solution [55, 56, 60–62].
iv. For finite domains, long distance particle interactions, from which the nonlocal constitutive behavior of the material originates, are in some way impeded by the body's boundary surface with consequent boundary effects. For instance, the redistribution of the source Hookean stress operated by the Eringen's fully nonlocal integral model is not complete and the non-redistributed fraction of the source local stress is simply lost out together with a fraction of the structural stiffness [37, 70].

Whereas the shortcomings listed at points i–iii have been widely discussed in the literature, the issue set down at point iv has received modest attention so far. In the present paper this issue is rediscussed and recognized to be a consequence of boundary effects arising from an intrinsic dispersive character of the model, whereby the boundary layer exhibits a compliance larger than that of the bulk material. An enhanced nonlocal integral model proposed by [70] (and independently by [71]) is here reconsidered for further developments and improvements, also with respect to a similar model studied by [68].

The proposed model is obtained by the addition of a suitable nonhomogeneous local phase to the Eringen's nonlocal model, such that it proves to be exempt from all the drawbacks exhibited by the Eringen's model, namely, it is equivalent to a Fredholm integral equation of the second kind at all points of the domain, and obeys the locality recovery condition, though in the milder form of *local stress recovery condition*. This means that, contrary to the strain-difference based model of [38] where both the stress and free energy lose their dependence on the internal length parameter in the presence of a uniform strain field, with the present model only the stress is required to possess this requisite.

For possible applications to micro- and nano-beams, the appropriate one-dimensional version of the proposed model is provided together with the associated sixth order differential form. This is equipped with the inherent nonlocality boundary conditions by which the derived differential equation—in contrast to the Eringen's fourth order differential equation—is fully equivalent to the associated integral equation. In this way, for every beam problem, at least in principle, two different solution methods are rendered available, one in the form of Fredholm integral equation of the second kind, the other in the form of sixth order differential equation with related ordinary and extra nonlocality boundary conditions (six boundary conditions in total), both of which lead to a same unique solution of the beam problem.

Only static problems will be addressed. Extensions to dynamics and buckling are possible, but not considered in this paper.

1.1 | Objectives

Within the context of homogeneous linear elasticity of solids, an enhanced nonlocal integral model is advanced, that is, a model whereby i) no dispersive boundary effects (with losses of stresses and stiffness) are allowed to occur, and ii) well-posed boundary-value problems are always dealt with. It generally predicts softening size effects without the most part of the anomalies mentioned before.

The main objective of the present paper is to study the behavior of small scale beams under static loads with particular concern to the inherent size effects. Specific objectives are the following:

a) To formulate the proposed integral model in the form of typical Fredholm integral equation of the second kind featured by a positive definite symmetric kernel, which guarantees the unique solvability of the structural problem [55, 56] and constitutes an improved form of the theory by [68].
b) To derive, aside the proposed nonlocal integral model, the equivalent sixth order differential equation together with the related two extra nonlocality boundary conditions, and to show that the differential approach to a beam problem leads to the same (unique) solution obtained through the integral approach.
c) To show that softening size effects are predicted for any type of beam and boundary conditions and that no paradoxes are allowed to occur.
d) To show that the values of the maximum deflection predicted by the proposed beam model are larger than those exhibited by the local model, and that they vary with the length scale parameter generally in a non-monotonic waved manner characterized by a limit asymptotic behavior like an atomic lattice model.

1.2 | Outline

After the Introduction, in Section 2 some boundary effects are discussed (larger compliance of the boundary layer, incomplete redistribution of Hookean stress with loss of stiffness). In Section 3 an enhanced nonlocal model is proposed, in which the mentioned boundary effects are eliminated with the introduction of a non-homogeneous local phase. The model's consistency is probed by a variational principle (principle of minimum total potential energy). In Section 4 the proposed model is particularized for Euler–Bernoulli beam models and a solution procedure with the integral approach is discussed. In Section 5 the extra nonlocality boundary conditions are presented. In Section 6 the sixth order differential equation for the enhanced nonlocal beam is derived and the solution procedure with the differential approach is discussed together with the related numerical algorithm. In Section 7 the results of several applications to benchmark beam problems are reported, graphically illustrated and discussed. In Section 8 the conclusions are drawn.

1.3 | Notation

Boldface symbols are used to denote vectors and tensors, with the usual contraction operations indicated by as many dots as the order of contraction. The symbol $:=$ denotes equality by definition. Other symbols will be provided in the text at their first appearance.

2 | BOUNDARY EFFECTS INDUCED BY THE ERINGEN'S NONLOCAL MODEL

2.1 | Generalities

For a homogeneous three-dimensional linear elastic solid, referred to orthogonal Cartesian co-ordinates $x_i, (i = 1, 2, 3)$, the Eringen's nonlocal elasticity model [1–4] can be cast in a standard form as

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V g_\ell(\mathbf{x}, \bar{\mathbf{x}})\mathbf{C} : \boldsymbol{\varepsilon}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{1}$$

Here, $\mathbf{C}$ is the (constant) elasticity tensor, $\boldsymbol{\sigma}(\mathbf{x})$ is the *nonlocal* stress field at point $\mathbf{x} \in V$, $\boldsymbol{\varepsilon}(\bar{\mathbf{x}})$ the corresponding *local* strain field at $\bar{\mathbf{x}} \in V$, whereas $g_\ell(\mathbf{x}, \bar{\mathbf{x}})$ is the kernel function. This is a two-point positive definite influence function of the form

$$g_\ell(\mathbf{x}, \bar{\mathbf{x}}) := g\left(-\frac{r}{\ell}\right) \tag{2}$$

where $r := |\mathbf{x} - \bar{\mathbf{x}}|$ and $\ell > 0$ is a length scale parameter.[1] Eringen [1, 2] suggested to take $\ell = e_0 a$ with a being a characteristic length of the microstructure (particle spacing, grain size, and the like), e_0 a non-dimensional material constant of value $e_0 \approx 0.391$. g_ℓ is assumed to satisfy the normalization condition

$$\int_{V_\infty} g_\ell(\mathbf{x}, \bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) = 1 \quad \forall \mathbf{x} \in V_\infty \tag{3}$$

where V_∞ denotes the infinite domain filled with the considered material [1, 3, 4, 8, 9]. Equation (3) implies that, for $\ell \to 0$, $g_\ell(\mathbf{x}, \bar{\mathbf{x}}) \to \delta_D(\mathbf{x}, \bar{\mathbf{x}})$ = Dirac delta, hence by (1) the classic Hooke law is recovered, whereas for $\ell \to \infty$, $g_\ell(\mathbf{x}, \bar{\mathbf{x}}) \to 0$, meaning that the fully nonlocal model (1) tends to behave as a perfectly soft model as ℓ tends to take larger values. The integral (3) may be physically interpreted as a normalized measure of the totality of particle interactions occurring around the generic point $\mathbf{x} \in V_\infty$; the unit value of this measure means that particle interaction is complete $\forall\, \mathbf{x} \in V_\infty$.

The kernel function (2) is in general a regular continuous function endowed with an unbounded support; it has its maximum for $r = 0$ and decreases more or less rapidly with increasing r, in such a way that $g_\ell \approx 0$ for $r > R$, where R (*conventional influence distance*, in contrast to the actual infinite one) is usually expressed as a multiple of ℓ, say $R = m\ell$ ($m \approx 6$ in the case of bi-exponential function).

For R sufficiently small with respect to the macroscopic dimension L of the body, the notion of *core domain*, say $V_c \subseteq V$, can be envisaged [70] which is the collection of points $\mathbf{x} \in V$ whose distance from ∂V is larger than R. Since for $\ell = 0$ it is $R = 0$, obviously $V_c = V$ correspondingly, whereas $V_c \subset V$ for $R < R^*$, but $V_c = \oslash$ for $R > R^* = m\ell^*$. The characteristic value R^* coincides with the radius of the largest spherical support (or conventional influence domain) entirely contained within the body's boundary surface.

For a given body of finite domain and a given kernel g_ℓ of infinite support, the function $\gamma_\ell(\mathbf{x})$ defined as

$$\gamma_\ell(\mathbf{x}) := \int_V g_\ell(\mathbf{x}, \bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{4}$$

expresses a normalized measure of the particle interactions (with consequent boundary effects) occurring around $\mathbf{x}$ within V. Due to (3) it is $\gamma_\ell(\mathbf{x}) < 1\, \forall \mathbf{x} \in V$, but $\gamma_\ell(\mathbf{x}) \approx 1$ at all points $\mathbf{x} \in V_c$, provided V_c be nonempty (i.e. $R < R^*$, or equivalently $\ell < \ell^*$). More details on this issue can be found in [70]. In Figure 1 the function $1 - \gamma_\ell(r)$ is plotted as a function of r/L for different values of $\lambda = \ell/L$ considering the case of the 1D bi-exponential kernel, that is, $g_\ell = \frac{1}{2\ell}\exp(-\frac{r}{\ell})$, in which case $R^* = 6\ell^* = 0.5L$, hence $\lambda^* = \frac{1}{12} \approx 0.08333$.

[1] Polizzotto (2001) [8] proposed to adopt the kernel function in the form $g_\ell(x, \bar{x}) = \bar{g}(-r_{\mathrm{eq}}/\ell_0)$, where ℓ_0 is the reference length scale parameter taken equal to the largest value of the space variable one, $\ell(\mathbf{x})$; r_{eq} is the *equivalent distance*, i.e., $r_{\mathrm{eq}} := r + r^*$, sum of the so-called *geodetical distance*, (length of the shortest path between every two points of the domain without intersecting its boundary) and the fictitious distance $r^* \geq 0$ accounting for the additional attenuation effects due to inhomogeneities. In general $r \geq |\mathbf{x} - \bar{\mathbf{x}}|$, but for a convex domain (no holes or cracks, nor rientrant angles at the boundary) it is $r = |\mathbf{x} - \bar{\mathbf{x}}|$. For simplicity, in the present paper r indicates the Euclidean distance.

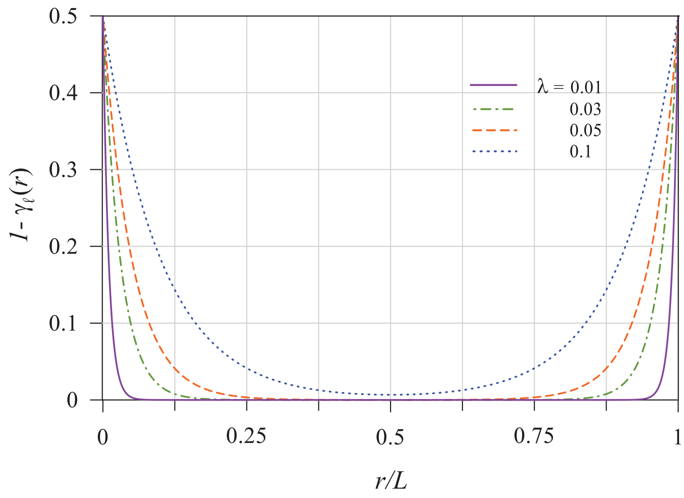

FIGURE 1 Function $1-\gamma_\ell(r)$ (*residual locality density*) plotted as a function of r/L for $\lambda = 0.01$ (solid line), 0.03 (dash-dot line), 0.05 (dashed line), 0.1 (dotted line). For $\lambda < \lambda^* = 0.08333$ a core domain of length $L_c = L(1-12\lambda) > 0$ exists where $1-\gamma_\ell(r) \approx 0$

2.2 | Boundary effects

The nonlocal stress which by (1) corresponds to a uniform local strain, say $\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}_0$=const., is given by

$$\boldsymbol{\sigma}(\mathbf{x}) = \gamma_\ell(\mathbf{x})\mathbf{C} : \boldsymbol{\varepsilon}_0 \tag{5}$$

Notably, the nonlocal stress (5) is not uniform as expected, it being expressed by a modified Hooke law encompassing a fictitious non-homogeneous moduli tensor $\mathbf{C}_f$ as

$$\mathbf{C}_f := \gamma_\ell(\mathbf{x})\mathbf{C} \tag{6}$$

Since, for $R < R^*$, $\gamma_\ell \approx 1$ within the core domain, whereas $\gamma_\ell < 1$ at points close to the boundary surface, it follows that, correspondingly, a boundary effect occurs whereby the local compliance measured at points located within the boundary layer (circumventing the core domain) increases the more, the closer is the point to the boundary surface $S = \partial V$.

As reported by [69], another manifestation of boundary effects is the incompleteness of the redistribution of the source local Hookean stress, say $\mathbf{s}(\bar{\mathbf{x}}) := \mathbf{C} : \boldsymbol{\varepsilon}(\bar{\mathbf{x}})$, at points within a finite domain. This issue is here briefly reported using a different argument. Namely, let Equation (1) be integrated over the domain V obtaining the equality

$$\int_V \boldsymbol{\sigma}(\mathbf{x})\,\mathrm{d}V(\mathbf{x}) = \int_V \mathbf{s}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) - \int_V [1-\gamma_\ell(\bar{\mathbf{x}})]\mathbf{s}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{7}$$

For an unbounded domain, say V_∞, in which it is $\gamma_\ell(\bar{\mathbf{x}}) = 1\ \forall\bar{\mathbf{x}} \in V_\infty$, Equation (7) simplifies as

$$\int_V \boldsymbol{\sigma}(\mathbf{x})\,\mathrm{d}V(\mathbf{x}) = \int_V \mathbf{s}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{8}$$

which means that the nonlocal stress $\boldsymbol{\sigma}(\mathbf{x})$ is the result of a complete redistribution of $\mathbf{s}(\bar{\mathbf{x}})$ within V_∞. Instead, in the case of finite domain $V \subset V_\infty$, it is $\gamma_\ell(\bar{\mathbf{x}}) < 1\ \forall\bar{\mathbf{x}} \in V$, but $\gamma_\ell(\bar{\mathbf{x}}) \approx 1$ within the core domain V_c, provided that $V_c \neq \emptyset$ (i.e., $R < R^*$). Therefore, for a finite domain, the stress redistribution is always incomplete at every point of the boundary layer, but it is almost complete at points within the core domain whenever this is nonempty.

We conclude this section stating that, for a body of finite domain, the Eringen's nonlocal model exhibits boundary effects consisting in an increased compliance of the boundary layer, along with an incomplete redistribution of the source Hooke stress, all of which encompass a loss of structural stiffness. This dispersive character of the nonlocal integral model is featured by the function $1-\gamma_\ell(\mathbf{x}) \geq 0$, which can be interpreted as a normalized measure of the residual local behavior of the generic particle within V, hence $1-\gamma_\ell(\mathbf{x})$ may be referred to as the *residual locality density* of the nonlocal body.

3 | ENHANCED FORM OF THE ERINGEN'S NONLOCAL INTEGRAL MODEL

In this section the formulation of a nonlocal integral continuum theory is presented, which is an enhanced form of the Eringen's fully nonlocal one.

3.1 | Formulation

A nonlocal model previously proposed by [70] is here reconsidered, whose constitutive equation is expressed in two alternative equivalent forms, namely,

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x}) + \int_V g_\ell(\mathbf{x},\bar{\mathbf{x}})\mathbf{C} : [\boldsymbol{\varepsilon}(\bar{\mathbf{x}}) - \boldsymbol{\varepsilon}(\mathbf{x})]\,\mathrm{d}V(\bar{\mathbf{x}}) = [1-\gamma_\ell(\mathbf{x})]\mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x}) + \int_V g_\ell(\mathbf{x},\bar{\mathbf{x}})\mathbf{C} : \boldsymbol{\varepsilon}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{9}$$

Another equivalent form of this model was independently proposed by [71] within damage mechanics, namely,

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V [g_\ell(\mathbf{x},\bar{\mathbf{x}}) + (1-\gamma_\ell(\mathbf{x}))\delta_D(\mathbf{x}-\bar{\mathbf{x}})]\mathbf{C} : \boldsymbol{\varepsilon}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{10}$$

which was subsequently used by [68] as a basis of a modified kernel theory. Equation $(9)_1$ was employed by [37, 38] as a basis of a strain-difference driven nonlocal continuum theory, whereas Equation $(9)_2$ is here reconsidered but cast in a more general form as

$$\boldsymbol{\sigma}(\mathbf{x}) = \xi(\mathbf{x})\mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x}) + \int_V g_\ell(\mathbf{x},\bar{\mathbf{x}})\mathbf{C} : \boldsymbol{\varepsilon}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{11}$$

that is, by adding to the r.h.s. of (1) a regularizing non-homogeneous local phase of the form $\xi(\mathbf{x})\mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x})$ with density $\xi(\mathbf{x}) > 0\ \forall\mathbf{x} \in V$. On choosing, as previously suggested by [70] and [37], the density $\xi(\mathbf{x}) = 1-\gamma_\ell(\mathbf{x})$, the residual local behavior of the nonlocal material is recovered. But this choice may lead to computational drawbacks as hereafter explained.

For infinite support kernel (e.g., the bi-exponential function), the condition $\xi(\mathbf{x}) = 1-\gamma_\ell(\mathbf{x}) > 0\ \forall\mathbf{x} \in V$ is satisfied, therefore Equation (11) correspondingly leads to a Fredholm integral equation of the second kind in the whole domain, at least in principle. However, for values of $\ell < \ell^*$, hence $R < R^*$, a non-empty core domain $V_c \subset V$ exists within which it is $1-\gamma_\ell(\mathbf{x}) \approx 0$, consequently the regularization role played by the added local phase may become ineffective within V_c for small values of ℓ, such that the numerical solution of the mentioned integral equation may hardly be accomplished without numerical trubles. For the above reasons, the density $\xi(\mathbf{x})$ is here chosen in the form

$$\xi(\mathbf{x}) := e + 1 - \gamma_\ell(\mathbf{x}) \tag{12}$$

with $e > 0$ being a (small) constant. With this choice, it is $\xi(\mathbf{x}) > e$ within the whole domain V, which makes Equation (11) lead to a Fredholm integral equation of the second kind at every point in V, including the core domain (if any) where $\xi(\mathbf{x}) \approx e > 0$ and therefore (11) always leads to well-posed boundary-value problems. The added homogeneous local phase $e\,\mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x})$ aims to dispense from numerical trubles while solving the governing integral equation—as a rule cast as a typical second kind Fredholm integral equation—for small values of ℓ. Due to the equivalence between $(9)_2$ and $(9)_1$, the proposed model also belongs to the family of strain-difference based models, but with the name "strain-difference based model" the theory advanced in [38] and [69] is intended. In Section 7 we shall return to this point.

The proposed model (11) and (12) can be recognized to be exempt from almost all the shortcomings of the original fully nonlocal model. For instance taking $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 = \text{const.}$, by (11) we get $\boldsymbol{\sigma}(\mathbf{x}) = (e+1)\mathbf{C} : \boldsymbol{\varepsilon}_0$. The proposed model also constitutes a good analytical tool for the analysis of size effects of structures at micro- and nano-scales; it generally predicts a softening behavior of the specimen with increasing the length scale parameter. As we shall better discuss in Section 7, the proposed model is in contrast to the strain gradient elastic model and to the strain-difference based nonlocal elastic one, both of which in fact predict stiffening size effects. In the next subsection, a variational formulation of the proposed model is presented.

3.2 | Principle of the minimum total potential energy

Let us consider a solid of volume V and boundary surface $S = \partial V$ with a nonlocal elastic material as described previously, featured by Equations (11) and (12). It is subjected to body forces, say $\mathbf{b}$, in V, surface traction, say $\mathbf{t}$, on $S_f \subset S$, and assigned displacements, say $\bar{\mathbf{u}}$, in $S_c = S \setminus S_f$. It can be shown that the relevant boundary-value problem admits variational formulations. Here the principle of the minimum total potential energy is reported. This consists in the minimization of the functional

$$W[\mathbf{u}] := \frac{1}{2}\int_V \int_V \boldsymbol{\varepsilon}(\mathbf{x}) : \mathbf{C}G(\mathbf{x},\mathbf{x}') : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x})\,\mathrm{d}V(\mathbf{x}') - \int_V \mathbf{b}\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}V(\mathbf{x}) - \int_{S_f} \mathbf{t}\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \tag{13}$$

where $\mathbf{u}$ denotes the displacement, $\boldsymbol{\varepsilon}$ the associated strain, whereas $G(\mathbf{x},\mathbf{x}')$ is a positive definite two-point function defined as[2]

$$G(\mathbf{x},\mathbf{x}') := \left\{e + 1 - \frac{1}{2}\left[\gamma_\ell(\mathbf{x}) + \gamma_\ell(\mathbf{x}')\right]\right\}\delta_D(\mathbf{x}-\mathbf{x}') + g_\ell\left(-\frac{|\mathbf{x}-\mathbf{x}'|}{\ell}\right) \tag{14}$$

The minimization of $W[\mathbf{u}]$ is subjected to the usual continuity requirements for $\mathbf{u}$ and compatibility for $\boldsymbol{\varepsilon}$ and $\mathbf{u}$, along with the constraint $\mathbf{u} = \bar{\mathbf{u}}$ on $S_c := S \setminus S_f$. Through a standard procedure, the first variation of (13) can be found to read, denoting virtual state variables by an upper tilde, as

$$\delta W = \int_V \nabla^s\tilde{\mathbf{u}}(\mathbf{x}) : \int_V G(\mathbf{x},\mathbf{x}')\mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') - \int_V \mathbf{b}(\mathbf{x})\cdot\tilde{\mathbf{u}}(\mathbf{x})\,\mathrm{d}V - \int_{S_f} \mathbf{t}(\mathbf{x})\cdot\tilde{\mathbf{u}}(\mathbf{x})\,\mathrm{d}S = 0 \tag{15}$$

With the notation

$$\boldsymbol{\sigma}(\mathbf{x}) := \int_V G(\mathbf{x},\mathbf{x}')\mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \tag{16}$$

and applying the divergence theorem, also noting that $\tilde{\mathbf{u}} = \mathbf{0}$ on S_c, (15) can be rewritten as

$$\delta W = -\int_V (\nabla\cdot\boldsymbol{\sigma} + \mathbf{b})\cdot\tilde{\mathbf{u}}\,\mathrm{d}V - \int_{S_f}(\boldsymbol{\sigma}\cdot\mathbf{n} - \mathbf{t})\cdot\tilde{\mathbf{u}}\,\mathrm{d}S = 0 \tag{17}$$

This equality implies the validity of the equilibrium equations, that is,

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V, \qquad \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{t} \quad \text{on } S_f \tag{18}$$

where $\boldsymbol{\sigma}$ coincides with the stress given by (16) and by (11).

[2] The function $G(\mathbf{x},\mathbf{x}')$ of (14) is a counterpart of the function $A_{mod}(\mathbf{x},\mathbf{x}')$ introduced by [68] with their Equation (19), as well as of the function $W(\mathbf{x},\mathbf{x}')$ introduced by [71] with their Equation (8) (from which $A_{mod}(\mathbf{x},\mathbf{x}')$ was borrowed). However, $G(\mathbf{x},\mathbf{x}')$ differs in two points, namely: (i) $G(\mathbf{x},\mathbf{x}')$ is fully symmetric, i.e., $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$; (ii) $G(\mathbf{x},\mathbf{x}')$ contains an additive constant e which makes the enhanced kernel lead to a Fredholm integral equation of the second kind everywhere in V, even within the core domain V_c where $G(\mathbf{x},\mathbf{x}') = e\delta_D(\mathbf{x}-\mathbf{x}') + g_\ell(-\frac{|\mathbf{x}-\mathbf{x}'|}{\ell})$ $\forall\mathbf{x}\in V_c$.

It thus results that the stationarity condition of W leads to the/a solution of the boundary-value problem under consideration and viceversa. Additionally, since the second variation of W is positive definite, that is,

$$\delta^2 W = \int_V \int_V \tilde{\boldsymbol{\varepsilon}}(\mathbf{x}) : \mathbf{C}G(\mathbf{x},\mathbf{x}') : \tilde{\boldsymbol{\varepsilon}}(\mathbf{x})' \,\mathrm{d}V(\mathbf{x})\,\mathrm{d}V(\mathbf{x}') > 0 \tag{19}$$

for arbitrary not trivially vanishing $\tilde{\boldsymbol{\varepsilon}}$, it results that W has a minimum at the solution, which therefore is unique. This variational principle may be useful for finite element discretizations; it is an extension of an analogous principle given by [8].

4 | ENHANCED NONLOCAL INTEGRAL MODEL FOR BEAMS

In this section, the previously proposed nonlocal integral model is applied to beam structures. Also, a solution method to solve the governing integral equation, previously used by [69], is extended to the present model with some novel additions. The extra boundary conditions induced by nonlocality and the inherent equivalent differential equation will be addressed in the two next sections.

4.1 | Generalities for beams in bending

Let us consider a beam of length L and constant cross section, referred to orthogonal Cartesian axes x, y, z, with x coincident with the centroid axis, z along the beam thickness, y along the width; the z axis is a principal axis of the cross section. The plane (x, z) coincides with the bending plane; the beam ends have co-ordinates $x = 0$ and $x = L$.

Considering the beam as an Euler–Bernoulli (EB) model in bending, and denoting $w(x)$ the beam transverse displacement of the centroid points (deflection), the local displacements can be cast with a standard notation as

$$u_x(x,z) = -z\,w'(x), \quad u_y(x,z) \equiv 0, \quad u_z(x,z) = w(x) \tag{20}$$

where $w'(x) = \mathrm{d}w/\mathrm{d}x$ is the (clockwise) rotation of the cross section at x. Therefore, the relevant meaningful strain component ε_{xx} is expressed as

$$\varepsilon_{xx}(x,z) = z\chi(x) \tag{21}$$

where $\chi(x) = -w''$ is the beam bending curvature. The related local Hooke stress, say $s_{xx}(x,z)$, is given by

$$s_{xx}(x,z) = Ez\chi(x) \tag{22}$$

where E denotes the Young modulus. Also, denoting by $\sigma_{xx}(x,z)$ the nonlocal (equilibrium) stress, the bending moment $M(x)$ is

$$M(x) = \int_A z\sigma_{xx}(x,z)\,\mathrm{d}A \tag{23}$$

Next, let (1) be specialized for the one-dimensional model under consideration, and let us assume that nonlocality effects do not propagate into the transverse direction, such that the kernel varies only in the x direction. Then, by an integration over the cross section, we easily arrive at the equation

$$M(x) = \int_0^L g_\ell(x,\bar{x}) D\chi(\bar{x})\,\mathrm{d}\bar{x} \tag{24}$$

where $D := EI$ with I being the second area moment of the cross section. Equation (24) is the Eringen's fully nonlocal integral equation for a beam model, that is, the equivalent of (1) for a beam, whereas the enhanced nonlocal integral equations for beams, corresponding to (11), can be written in the form

$$M(x) = \xi(x) D\chi(x) + \int_0^L g_\ell(x, \bar{x}) D\chi(\bar{x})\,\mathrm{d}\bar{x} \tag{25}$$

where $\xi(x)$ is the counterpart of (12), that is

$$\xi(x) := e + 1 - \gamma_\ell(x) \quad \forall x \in (0, L) \tag{26}$$

The bending moment $M(x)$ has to satisfy the equilibrium equation $M''(x) + p(x) = 0$, where $p(x)$ is a given distributed load.

Following a procedure similar to the one adopted by [69] and by [72], let the set of equilibrium bending moments be represented in the form

$$M(x) = -f(x) - \frac{D}{L^2}(C_1 x + C_2 L) \tag{27}$$

where C_1 and C_2 are arbitrary (non-dimensional) constants, whereas $f(x)$ is a particular function such that $f''(x) = p(x)$ everywhere in $(0, L)$, which is here taken as

$$f(x) := \int_0^x (x - \bar{x}) p(\bar{x})\,\mathrm{d}\bar{x} \tag{28}$$

It is useful to note that the bending moment (27) is the superposition of three bending moment functions, say

$$M(x) = M_0(x) + C_1 M_1(x) + C_2 M_2(x) \tag{29}$$

where $M_0(x)$, $M_1(x)$, and $M_2(x)$ denote the bending moments of an *auxiliary statically determinate* beam of length L. According to (27), the mentioned auxiliary beam is a cantilever beam clamped at $x = L$, free at $x = 0$, subjected to three *basic load conditions*, that is, the given load $p(x)$, a (downward) concentrated force of intensity D/L^2 at the free end, and an (anticlockwise) concentrated couple of intensity D/L at the free end. The bending moments of the auxiliary beam are

$$\left.\begin{aligned} M_0(x) &= -\int_0^x (x - \bar{x}) p(\bar{x})\,\mathrm{d}\bar{x} = -f(x) \\ M_1(x) &= -Dx/L^2, \quad M_2(x) = -D/L \end{aligned}\right\} \tag{30}$$

satisfying the conditions $M_0''(x) = -p(x)$ and $M_1''(x) = M_2''(x) = 0$ at every point along the beam.

One may choose the auxiliary beam differently constrained, in which case the basic bending moments M_0, M_1, M_2 change somewhat, but anyway $M_1(x)$ must be a linear function, $M_2(x)$ a constant one. However, this option is useless because irrelevant for the response of the actual beam.

4.2 | Auxiliary integral equations

As shown hereafter, the integral equation (25) is equivalent to a set of three mutually independent Fredholm integral equations of the second kind. Substituting (29) into (25) and making use of the function $\xi(x)$ of (26), Equation (25) can be written as

$$\xi(x)\,\chi(x) + \int_0^L g_\ell(x,\bar{x})\chi(\bar{x})\,\mathrm{d}\bar{x} = \frac{1}{D}[M_0(x) + C_1 M_1(x) + C_2 M_2(x)] \tag{31}$$

where M_0, M_1, M_2 are given by (30).

Next, let the unknown curvature function $\chi(x)$ be decomposed as

$$\chi(x) = \chi_0(x) + C_1\chi_1(x) + C_2\chi_2(x) \tag{32}$$

where C_1, C_2 are the *same constants* appearing in (27) and (30), whereas $\chi_n(x)$, $(n = 0, 1, 2)$, are some (unknown) *auxiliary curvatures*. Substituting (32) into (31) and with a re-ordering gives the equality

$$\xi(x)\,\chi_0(x) + \int_0^L g_\ell(x,\bar{x})\chi_0(\bar{x})\,\mathrm{d}\bar{x} - \frac{1}{D}M_0(x) + C_1\left[\xi(x)\,\chi_1(x) + \int_0^L g_\ell(x,\bar{x})\chi_1(\bar{x})\,\mathrm{d}\bar{x} - \frac{1}{D}M_1(x)\right]$$
$$+ C_2\left[\xi(x)\,\chi_2(x) + \int_0^L g_\ell(x,\bar{x})\chi_2(\bar{x})\,\mathrm{d}\bar{x} - \frac{1}{D}M_2(x)\right] = 0 \tag{33}$$

Since C_1 and C_2 are arbitrary constants, we can write

$$\xi(x)\chi_n(x) + \int_0^L g_\ell(x,\bar{x})\chi_n(\bar{x})\,\mathrm{d}\bar{x} = \frac{1}{D}M_n(x) \quad (n = 0, 1, 2) \tag{34}$$

This relation constitutes a set of three mutually independent integral equations in the unknown auxiliary curvatures χ_n. These latter equations govern, respectively, the deformation of the auxiliary beam subjected to every one of the three basic load conditions mentioned before. Furthermore, these integral equations can be transformed such as they take on each the typical form of a Fredholm integral equation of the second kind [55]. For this purpose, since $\xi(x) > e > 0\ \forall x \in (0, L)$, let us posit

$$\left.\begin{aligned} K(x,\bar{x}) &:= g_\ell(x,\bar{x})/\sqrt{\xi(x)\xi(\bar{x})} \\ \psi_n(x) &:= L\sqrt{\xi(x)}\,\chi_n(x),\ (n = 0, 1, 2) \end{aligned}\right\} \tag{35}$$

where the ψ_n are non-dimensional variables replacing χ_n. With this notation in mind, (34) can be rewritten as

$$\boxed{\psi_n(x) + \int_0^L K(x,\bar{x})\psi_n(\bar{x})\,\mathrm{d}\bar{x} = F_n(x) \quad (n = 0, 1, 2)} \tag{36}$$

where

$$F_n(x) := \begin{cases} \dfrac{L}{D}\dfrac{M_0(x)}{\sqrt{\xi(x)}} & \text{for } n = 0 \\[2ex] \dfrac{L}{D}\dfrac{M_1}{\sqrt{\xi(x)}} & \text{for } n = 1 \\[2ex] \dfrac{L}{D}\dfrac{M_2}{\sqrt{\xi(x)}} & \text{for } n = 2 \end{cases} \tag{37}$$

Equations (36) and (37) constitute a set of three uncoupled typical Fredholm integral equations of the second kind each with a symmetric (positive definite) kernel $K(x,\bar{x})$. Every integral equation admits a unique solution [55] and can be solved numerically through a routine computational method (see e.g. [73]).

4.3 | Solution procedure for the nonlocal integral problem

The uncoupled equations (36) are independent of the way the given beam is actually constrained; also, only the first one ($n = 0$) is influenced by the actual load. This implies that the second and third equations (36) can be solved preliminarily, once for all load cases to be treated.

Therefore, once the functions $\psi_n(x)$, ($n = 0, 1, 2$), have been evaluated, then the deflection $w(x)$ of the beam can be obtained by integration of (32). Namely, recalling that $\chi(x) = -w''(x)$, by a double integration over $(0, L)$ we can rewrite (32) as

$$\boxed{w(x) = w_0(x) + C_1 w_1(x) + C_2 w_2(x) + C_3 x + C_4 L} \tag{38}$$

where C_3 and C_4 are further (non-dimensional) constants. The functions $w_n(x)$, ($n = 0, 1, 2$), are expressed as

$$\boxed{w_n(x) := -\frac{1}{L}\int_0^x (x-\bar{x})\frac{\psi_n(\bar{x})}{\sqrt{\xi(\bar{x})}}\,\mathrm{d}\bar{x}, \quad (n = 0, 1, 2)} \tag{39}$$

and satisfy the equations $w_n''(x) = -\chi_n(x), (n = 0, 1, 2)$, hence they possess the meaning of *auxiliary deflections* of the auxiliary beam subjected to the three basic loads, respectively.[3]

In conclusion of the above, Equations (29) and (38) provide a closed-form representation of the response functions of the beam in which only the constants C_1, C_2, C_3, C_4 are unknown. These must be determined by means of the ordinary boundary conditions associated to the actual beam problem. All this will be better explained in the sequel with the applications.

5 | BOUNDARY CONDITIONS INDUCED BY NONLOCALITY

The solution (M, χ) of (25) satisfies naturally some extra boundary conditions which we now go to determine for subsequent use within the Eringen's integro-differential approach to nonlocal elastic beams. In view of this task, the kernel function is taken in the form of a bi-exponential function, namely,

$$g_\ell(x,\bar{x}) = \frac{1}{2\ell}\exp\left(-\frac{r}{\ell}\right), \quad r := |x-\bar{x}| \tag{40}$$

which was suggested by [2] as the Green function of the Helmholtz differential equation

$$M(x) - \ell^2 M''(x) = D\chi(x) \tag{41}$$

Following [56], let us take the first derivative of (25), that is,

$$M'(x) = \xi(x)D\chi'(x) - \gamma_\ell'(x)D\chi(x) - \frac{1}{\ell}\int_0^L \frac{1}{2\ell}\exp\left(-\frac{r}{\ell}\right)\frac{x-\bar{x}}{r}D\chi(\bar{x})\,\mathrm{d}\bar{x} \tag{42}$$

[3] Equation (39) implies that $w_n(0) \neq 0$ and $w_n'(0) \neq 0$, instead of $w_n(L) = 0$ and $w_n'(L) = 0$ as required by the auxiliary cantilever beam clamped at $x = L$. This contradiction is without consequences for the response of the actual beam, for it will be compensated through the values of the constants C_3 and C_4. If one wishes, the apparent contradiction may be easily eliminated.

where we have used the equality

$$\frac{\mathrm{d}}{\mathrm{d}x}g\left(-\frac{r}{\ell}\right)=-\frac{1}{\ell}g\left(-\frac{r}{\ell}\right)\frac{x-\bar{x}}{r} \tag{43}$$

The following equalities are also reported for later use, i.e.,

$$\gamma_\ell(x)=\int_0^L \frac{1}{2\ell}\exp\left(-\frac{r}{\ell}\right)\mathrm{d}\bar{x}=1-\frac{1}{2}\left[\exp\left(-\frac{x}{\ell}\right)+\exp\left(-\frac{L-x}{\ell}\right)\right] \tag{44}$$

and

$$\gamma_\ell'(x)=-\frac{1}{\ell}\int_0^L \frac{1}{2\ell}\exp\left(-\frac{r}{\ell}\right)\frac{x-\bar{x}}{r}\,\mathrm{d}\bar{x}=\frac{1}{2\ell}\left[\exp\left(-\frac{x}{\ell}\right)-\exp\left(-\frac{L-x}{\ell}\right)\right] \tag{45}$$

from which we have

$$\gamma_\ell(0)=\gamma_\ell(L)=k_\ell:=\frac{1}{2}\left[1-\exp\left(-\frac{L}{\ell}\right)\right] \tag{46}$$

and

$$\gamma_\ell'(0)=\frac{k_\ell}{\ell},\qquad \gamma_\ell'(L)=-\frac{k_\ell}{\ell} \tag{47}$$

Next, let us particularize (25) and (42) to obtain $M(x)$ and $M'(x)$ at $x=0$, namely, recalling (40) and (42),

$$\left.\begin{aligned}M(0)&=\xi(0)D\chi(0)+\int_0^L \frac{1}{2\ell}\exp\left(-\frac{|0-\bar{x}|}{\ell}\right)D\chi(\bar{x})\mathrm{d}\bar{x}\\ M'(0)&=\xi(0)D\chi'(0)-\frac{k_\ell}{\ell}D\chi(0)+\frac{1}{\ell}\int_0^L \frac{1}{2\ell}\exp\left(-\frac{|0-\bar{x}|}{\ell}\right)D\chi(\bar{x})\,\mathrm{d}\bar{x}\end{aligned}\right\} \tag{48}$$

Analogously, let us repeat the above operation at $x=L$ to get

$$\left.\begin{aligned}M(L)&=\xi(L)D\chi(L)+\int_0^L \frac{1}{2\ell}\exp\left(-\frac{L-\bar{x}}{\ell}\right)D\chi(\bar{x})\mathrm{d}\bar{x}\\ M'(L)&=\xi(L)D\chi'(L)+\frac{k_\ell}{\ell}D\chi(L)-\frac{1}{\ell}\int_0^L \frac{1}{2\ell}\exp\left(-\frac{L-\bar{x}}{\ell}\right)D\chi(\bar{x})\,\mathrm{d}\bar{x}\end{aligned}\right\} \tag{49}$$

Next, let us compute the quantities $M(0)-\ell M'(0)$ and $M(L)+\ell M'(L)$ using for this purpose (48) and (49). Then, observing that $\xi(0)=\xi(L)=e+1-k_\ell$, we obtain the desired extra boundary conditions as

$$\boxed{\begin{aligned}M(0)-\ell M'(0)&=(e+1)D\left[\chi(0)-c\ell\chi'(0)\right]\\ M(L)+\ell M'(L)&=(e+1)D\left[\chi(L)+c\ell\chi'(L)\right]\end{aligned}} \tag{50}$$

where the scalar c is defined as

$$c:=1-k_\ell/(e+1) \tag{51}$$

The equalities (50) are referred to with the name *extra nonlocality boundary conditions* (in contrast to the name *extra gradient boundary conditions* for gradient models).[4]

Let us note that (50) are certainly satisfied by a statically and kinematically admissible (SKA) solution $(M(x), \chi(x))$ of the integral equation (25) whenever the latter solution does exist as it is the case here. Equation (50) imposes cumulative restrictions on both the bending moment and the curvature χ, hence there remains sufficient freedom to accommodate the equilibrium conditions and the solution is allowed to exist, which implies that correspondingly the beam problem admits a solution. Conversely, if the set of SKA solutions $(M(x), \chi(x))$ satisfying (50) is empty, then the nonlocal beam problem cannot have solution. Therefore, the extra nonlocality boundary conditions (50) can be interpreted as *existence boundary conditions* for a solution of (25).

In the case of the original Eringen's nonlocal model (24) (with which (25) identifies itself on taking $\xi(x) \equiv 0$), the nonlocality boundary conditions simplify by taking the form known from [56], that is,

$$M(0) - \ell M'(0) = M(L) + \ell M'(L) = 0 \tag{52}$$

These impose restrictions on the bending moment M alone, which generally are in contrast with the equilibrium equations such as to impede the beam problem to admit a solution [62, 69].[5]

6 | EQUIVALENT DIFFERENTIAL EQUATION FOR ENHANCED NONLOCAL BEAMS

In this section the differential equation equivalent to the integral equation (25) with the kernel g_ℓ in the form (40) is determined.

The invoked concept of equivalence means that the desired differential equation leads to a solution coincident with that of the correspondent nonlocal integral equation. The equivalence is guaranteed if the nonlocality boundary conditions (50) are concomitantly satisfied [56].

6.1 | Equivalent differential equation

To this purpose, let the Helmholtz operator $\mathcal{L} := 1 - \ell^2 \left(\mathrm{d}^2/\mathrm{d}x^2\right)$ (of which the kernel (40) is the Green function) be applied to (25). Then, we obtain the desired equation in the form

$$\mathcal{L}[\xi(x)\chi(x)] + \chi(x) = \frac{1}{D}\mathcal{L}M(x) \tag{53}$$

The function $\xi(x)$, coinciding with (26), recalling (40) and (44), is equivalent to

$$\xi(x) = e + 1 - \gamma_\ell(x) = e + \frac{1}{2}\left[\exp\left(-\frac{x}{\ell}\right) + \exp\left(-\frac{L-x}{\ell}\right)\right] \tag{54}$$

Next, on differentiating (53) twice with respect to x, considering that $M_1''(x) = M_2''(x) \equiv 0$, and recalling that $\chi(x) = -w''(x)$, leads to the governing differential equation as

$$\boxed{\mathcal{L}\left[\xi(x)w''(x)\right]'' + w''''(x) = \frac{1}{D}\mathcal{L}p(x)} \tag{55}$$

This is a linear sixth order differential equation with variable coefficients, which governs the deflection $w(x)$ of the enhanced nonlocal elastic beam. Assuming $\xi(x) \equiv 0$, (55) simplifies as $w''''(x) = \mathcal{L}p(x)/D$, that is the fourth order differential equation associated to the Eringen's fully nonlocal integral equation is recovered [27].

[4] The nonlocality boundary conditions are often in the literature [60–62] named *constitutive boundary conditions*. Here we prefer to use a unified nomenclature to denote extra boundary conditions occurring both in nonlocal-differential and gradient theories.

[5] In the case of an infinite domain in which the elastic response is asymptotically vanishing at infinite, the nonlocality boundary conditions (52) are satisfied automatically. This implies that the Fredholm integral equation (of the first kind) may admit a solution since the equilibrium equations can then be freely accommodated.

For the integration of the differential equation (55) the four ordinary boundary condition of the specific beam problem plus the two nonlocality boundary conditions (50) must be considered.

6.2 | Solution scheme for the equivalent differential equation (55)

A solution scheme of the enhanced nonlocal beam problem through the differential approach is addressed in this subsection.

6.2.1 | Auxiliary differential equations

For this purpose, recalling (29), let the differential equation (53) be rewritten in the form

$$\mathcal{L}[\xi(x)\chi(x)] + \chi(x) = \frac{1}{D}\mathcal{L}\left[M_0(x) + C_1 M_1(x) + C_2 M_2(x)\right] \tag{56}$$

Also, let the curvature χ be split as in (32), such that (56) becomes

$$\begin{aligned}&\left\{\mathcal{L}\left[\xi(x)\chi_0(x)\right] + \chi_0(x) - \frac{1}{D}\mathcal{L}M_0(x)\right\} + C_1\left\{\mathcal{L}\left[\xi(x)\chi_1(x)\right] + \chi_1(x) - \frac{1}{D}\mathcal{L}M_1(x)\right\}\\ &+C_2\left\{\mathcal{L}\left[\xi(x)\chi_2(x)\right] + \chi_2(x) - \frac{1}{D}\mathcal{L}M_2(x)\right\} = 0\end{aligned} \tag{57}$$

Since the latter equation has to hold for arbitrary values of C_1 and C_2, it splits into three uncoupled differential equations, namely,

$$\boxed{\mathcal{L}\left[\xi(x)\chi_n(x)\right] + \chi_n(x) = \frac{1}{L}\Phi_n(x), \quad (n = 0, 1, 2)} \tag{58}$$

where the (non-dimensional) functions $\Phi_n(x)$, recalling (30), can be defined as

$$\Phi_n(x) := \begin{cases}(L/D)\mathcal{L}M_0(x) = -L[f(x) - \ell^2 p(x)]/D & \text{for } n = 0\\ (L/D)M_1(x) = -x/L & \text{for } n = 1\\ (L/D)M_2(x) = -1 & \text{for } n = 2\end{cases} \tag{59}$$

The functions $\chi_n(x), n = (0, 1, 2)$, can therefore be referred to as the *auxiliary curvatures*, that is, the curvatures of the auxiliary beams subjected to the basic load conditions, respectively.

The boundary conditions associated to the second order differential equations (59) are those particular nonlocality boundary conditions that can be derived from (50) by there substituting M and χ with (29) and (32), respectively. (No ordinary boundary conditions are required for the auxiliary beams.) We obtain the relations

$$\boxed{\begin{aligned}&\chi_n(0) - c\,\ell\,\chi_n'(0) = P_n/L := [M_n(0) - \ell M_n'(0)]/[(e+1)D]\\ &\chi_n(L) + c\,\ell\,\chi_n'(L) = Q_n/L := [M_n(L) + \ell M_n'(L)]/[(e+1)D]\end{aligned}}\ (n = 0, 1, 2) \tag{60}$$

where c is given by (51). Also, recalling (30), the constants P_n, Q_n can be found to be expressed as

$$\left.\begin{aligned}&P_0 = 0, \quad P_1 = \frac{\ell/L}{e+1}, \quad P_2 = -\frac{1}{e+1}\\ &Q_0 = -\frac{f(L)+\ell f'(L)}{(e+1)D}, \ Q_1 = -\frac{1+\ell/L}{e+1};\ Q_2 = -\frac{1}{e+1}\end{aligned}\right\} \tag{61}$$

Equation (60) gives the extra nonlocality boundary conditions for the auxiliary beams under the three basic loads introduced before.

6.2.2 | Evaluation of the deflection function of the given beam

Equations (58) and (60) constitute, for every $n = (0, 1, 2)$, a second order differential equation in the unknown χ_n accompanied by two nonlocality boundary conditions, every equation being independent of the others. Integration of these auxiliary differential equations constitute a first step of the integration scheme for the beam problem under study which will be discussed shortly.

The second step consists in the integration of the equation

$$-w''(x) = \chi(x) = \chi_0(x) + C_1\chi_1(x) + C_2\chi_2(x) \tag{62}$$

where (32) has been used. Therefore, the deflection function $w(x)$ can be cast in the form

$$\boxed{w(x) = w_0(x) + C_1 w_1(x) + C_2 w_2(x) + C_3 x + C_4 L} \tag{63}$$

where C_3, C_4 are two further (non-dimensional) constants. The auxiliary deflections $w_n(x), (n = 0, 1, 2)$, are particular integrals of the equations $w_n''(x) = -\chi_n(x)$, which can be cast in the form

$$\boxed{w_n(x) = -\int_0^x (x - \bar{x})\chi_n(\bar{x})\,\mathrm{d}\bar{x}} \; (n = 0, 1, 2) \tag{64}$$

The constants C_1, C_2, C_3, C_4 must be determined by means of the four ordinary boundary conditions of the specific beam problem governed by Equations (58) and (63).

6.2.3 | Numerical algorithm for the auxiliary beam problems

The numerical algorithm used to address the auxiliary beam problems is presented in this section. For this purpose, let Equations (58) and (60) be first restated in more appropriate forms. On expanding the Helmholtz operator and by a reordering, the differential Equation (58) (multiplied by L) can be cast in the alternative form

$$[\xi(x) + 1]X(x) - \ell^2[\xi(x)X(x)]'' = \Phi(x) \tag{65}$$

where $X(x) := L\chi(x)$ (non-dimensional curvature) and the subscript n of χ_n and of Φ_n has been omitted for simplicity of notation.

Next, let us introduce the non-dimensional abscissa $s := x/L$ along with the new (non-dimensional) variable defined by the relation

$$y(s) := \xi(s)X(s) \tag{66}$$

where, for simplicity sake, $\xi(Ls)$ and $X(Ls)$ are rewritten as $\xi(s)$ and $X(s)$, respectively. Substituting (66) into (65), the latter equation can be rewritten in the equivalent form

$$\boxed{y''(s) = q(s)y(s) + d(s)} \tag{67}$$

where the primes denote derivatives with respect to s, whereas the coefficients $q(s)$ and $d(s)$ are expressed as

$$\left.\begin{aligned} q(s) &:= [\xi(s) + 1]/(\lambda^2\xi(s)) \\ d(s) &:= -\Phi(s)/\lambda^2 \end{aligned}\right\} \tag{68}$$

and $\lambda := \ell/L$ is the non-dimensional length scale parameter.

Correspondingly, recalling Equations (43)–(46) and (51), also noting that

$$\xi(0) = \xi(1) = (e+1)c, \quad \xi'(0) = -\xi'(1) = -k_\ell/\lambda \tag{69}$$

the boundary conditions (60) can be rewritten in the form (again omitting the n subscript)

$$\left.\begin{aligned} &\frac{y(0)}{\xi(0)} - c\lambda\left(\frac{y(s)}{\xi(s)}\right)'\Big|_{s=0} = P \\ &\frac{y(1)}{\xi(1)} + c\lambda\left(\frac{y(s)}{\xi(s)}\right)'\Big|_{s=1} = Q \end{aligned}\right\} \tag{70}$$

These, expanding the derivatives and reordering, finally become

$$\boxed{\begin{aligned} &y'(0) - \tfrac{1}{\lambda}y(0) = -\tfrac{e+1}{\lambda}P \\ &y'(1) + \tfrac{1}{\lambda}y(1) = \tfrac{e+1}{\lambda}Q \end{aligned}} \tag{71}$$

The numerical algorithm here used to address (67) and (71) is based on the *Stoemer's rule* (see [73], p. 726). This rule assumes that the given differential equation does not involve the first derivative of the unknown function and that the boundary conditions are *initial* boundary conditions. Whereas the first condition is satisfied in the present case, the second one is not, due to the evident mixed character exhibited by (71). For this reason the mentioned rule has been implemented here with some modifications whereby the solution is obtained as the solution of a system of linear algebraic equations.

Let the interval [0,1] be subdivided into m sub-intervals of equal length $h = 1/m$ and let $s_k, (k = 0, 1, \dots, m)$, denote the subdivision points. Then, through a simple finite difference method, let (67) be enforced at the points $s_k, k = (1, 2, \dots\dots, m-1)$, by writing

$$y_{k+1} - 2y_k + y_{k-1} = h^2(q_k y_k + d_k), \qquad (k = 1, 2, \dots, m-1) \tag{72}$$

where $q_k := q(s_k), d_k := d(s_k)$ and $y_k \approx y(s_k)$. At the end points $s_0 = 0$ and $s_m = 1$, using the notation $z_0 := y'(0)$ and $z_m := y'(1)$, we can write

$$\left.\begin{aligned} &y_1 - y_0 = hz_0 + \tfrac{1}{2}h^2(q_0 y_0 + d_0) \\ &y_m - y_{m-1} = hz_m - \tfrac{1}{2}h^2(q_m y_m + d_m) \end{aligned}\right\} \tag{73}$$

The unknown quantities z_0 and z_m can be derived from the boundary conditions (71), namely,

$$z_0 = \frac{1}{\lambda}y_0 - \frac{e+1}{\lambda}P, \qquad z_m = -\frac{1}{\lambda}y_m + \frac{e+1}{\lambda}Q \tag{74}$$

Therefore, substituting (74) into (73) and after reordering we can obtain

$$\left.\begin{aligned} y_1 - (1 + h/\lambda + \tfrac{1}{2}h^2 q_0)y_0 &= \tfrac{1}{2}h^2 d_0 - h(e+1)P/\lambda \\ y_{k-1} - (2 + h^2 q_k)y_k + y_{k+1} &= h^2 d_k, \quad (k = 1, 2, \dots, m-1) \\ (1 + h/\lambda + \tfrac{1}{2}h^2 q_m)y_m - y_{m-1} &= -\tfrac{1}{2}h^2 d_m + h(e+1)Q/\lambda \end{aligned}\right\} \tag{75}$$

Equation (75) is a system of $m+1$ linear algebraic equations with as many unknowns y_k. The solution of this equation system is a set of tabulated values y_k by which an interpolated function, say $y^h(s) \approx y(s)$ can be constructed.

The algorithm described above must be applied sequentially for $n = 0, 1, 2$ to obtain the non-dimensional auxiliary curvatures $X_n(s) = y_n(s)/\xi(s)$ with which the beam deflection (63) can be obtained as

$$w(s)/L = [w_0(s) + C_1 w_1(s) + C_2 w_2(s)]/L + C_3 s + C_4 \tag{76}$$

where

$$w_n(s)/L = -\int_0^s (s - \bar{s}) X_n(s)\, \mathrm{d}\bar{s}, \quad (n = 0, 1, 2) \tag{77}$$

7 | APPLICATIONS TO BEAMS

This section is devoted to applications to beams. The main concern is to show that the beam response functions can be equally obtained by either the integral approach presented in Section 4, or the differential approach presented in Section 6. The so obtained results are compared with analogous results obtained using other theories. For this purpose we considered the strain gradient elasticity theory [74–76] and the Eringen's nonlocal/differential elasticity theory [2, 27]. A comparison with the results given by [68] and by [69, 72] is also presented. The following beam cases and related (ordinary) boundary conditions have been considered:

a) Clamped-Free beam under point load P applied at the free end, with boundary conditions $w(0) = w'(0) = 0, M(L) = 0, M'(L) = P$ (CF1 case).
b) Clamped-Free beam under uniform distributed load p_0, with boundary conditions $w(0) = w'(0) = 0, M(L) = M'(L) = 0$ (CF2 case).
c) Pinned-Pinned beam under uniform distributed load p_0 with boundary conditions $w(0) = w(L) = M(0) = M(L) = 0$ (PP case).
d) Clamped-Pinned beam under uniform distributed load p_0 with boundary conditions $w(0) = w'(0) = 0, w(L) = M(L) = 0$ (CP case).
e) Clamped-Clamped beam under uniform distributed load p_0 with boundary conditions $w(0) = w'(0) = w(L) = w'(L) = 0$ (CC case).

7.1 | Solution procedure using the proposed theory

Every beam case was addressed by means of the proposed theory using both the integral and differential approaches. With the integral approach the auxiliary beam problems were solved first to obtain the auxiliary deflections (39). The response function (38), for every beam case, has then been obtained by enforcing the ordinary boundary conditions. With the differential approach the equivalent differential equation was solved for every auxiliary beam problem in terms of curvatures, using for this purpose the extra nonlocality boundary conditions (60), then the values of the remaining four constants were determined like with the previous procedure. The two approaches produced the same results, to within modest numerical approximations.

To illustrate this outcome, the normalized maximum deflection, say $\delta(\lambda)$, was computed using the two approaches for the CF1 and CP beam cases, and plotted in Figures 2(a,b) as a function of the non-dimensional length scale parameter $\lambda := \ell/L$, for different values of e, from $e = 10^{-1}$ to $e = 10^{-6}$. The normalization therein used makes that $\delta(0) = 1$, i.e., the maximum deflection of the fully local beam is taken equal to unit. These plots show that the two series of curves are almost overlapping; indeed, the two approaches are substantially alternative to each other. The guiding idea for the choice of the numerical value of e was, on the one hand, to operate with modest values of e such as to avoid a dominant regularizing local phase and, on the other hand, to avoid the occurrence of a possible numerical unstable behavior at minor values of e. Notably, numerically stable results were obtained on choosing values of e as small as 10^{-6}. Similar results were found for the other beam cases.

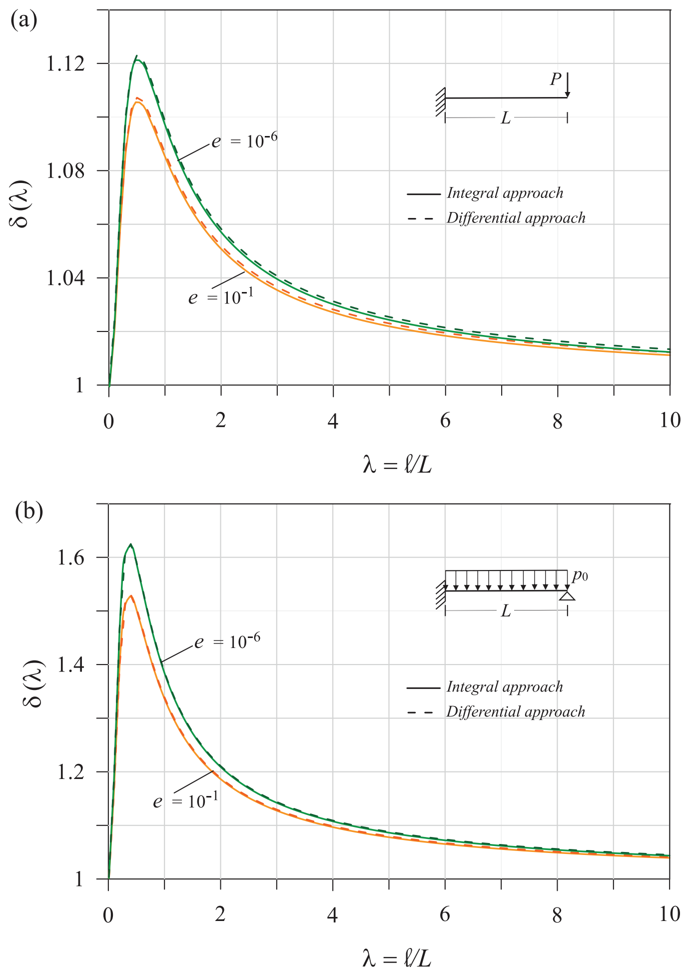

FIGURE 2 Normalized maximum deflection $\delta(\lambda)$ versus non-dimensional length scale parameter $\lambda = \ell/L$ for the enhanced beam model with fixed values of $e = 10^{-1}$ and $e = 10^{-6}$ using the integral approach (solid lines) and the differential approach (dashed lines): (a) CF1 case; (b) CP case

We also computed two series of outputs for every beam case.

i) The *normalized deflection* $w(x)/w_0$ (w_0 being the related local reference value) is plotted as a function of the adimensional abscissa x/L, each for different values of $\lambda(= 0, 0.5, 1., 1.5, 2.)$ and for a fixed value $e = 10^{-3}$.

These outputs are plotted in Figures 3(a-e). It is clearly shown that the proposed model predicts *softening size effects*, that is, the deformation corresponding to $\lambda \neq 0$ is larger than that corresponding to $\lambda = 0$, and this for all the considered beam cases. We found that the (non-dimensional) deflection curves are all contained within a strip comprised between the curve corresponding to $\lambda = 0$ as a lower bound and to that corresponding to $\lambda = 0.5$ as an upper bound. On increasing λ from $\lambda = 0$, the deflection curve raises up till a finite upper limit for $\lambda \approx 0.5$, then it lowers asymptotically towards the lower bound curve for $\lambda \to \infty$. No paradoxes, nor other shortcomings, occurred.

ii) The *normalized maximum deflection* $\delta(\lambda)$, as a function of λ, each for different values of $e(= 10^{-4}, 10^{-2}, 10^{-1}, 0.5)$.

These outputs are plotted in Figures 4(a-e) which show that, for every beam case, the maximum deflection δ exhibits a waved pattern with positive slopes for smaller values of λ, negative slopes for major values of λ, and with an intermediate value $\lambda \approx 0.5$ at which δ reaches a maximum (finite) value. The curves $\delta(\lambda)$ corresponding to $e = 0.5$ (dotted lines) indicate an expected notably less softening effect due to the higher value of e. The waved pattern of the curves $\delta(\lambda)$ has a counterpart in Figures 3(a-e) where all the curves $w(x)$ fall inside the strip comprised between the curve $w(x)$ corresponding to $\lambda \approx 0.50$ as an upper bound, and the curve obtained for $\lambda = 0$ as a lower bound. For all beam cases, the waved pattern of the curves $\delta(\lambda)$ is characterized by an asymptotic behavior whereby, for $\lambda \to \infty$, $\delta(\lambda) \to \delta_\infty$ with $\delta_\infty \approx 1$. It is worth noting that the waved pattern shown by the curve $\delta(\lambda)$ has a physical/mathematical motivation. Indeed, the governing integral equation of the beam problem, Equation (31), tends to become $(1 + e)\chi(x) = M/D$ for either $\ell \to 0$ and $\ell \to \infty$. Therefore,

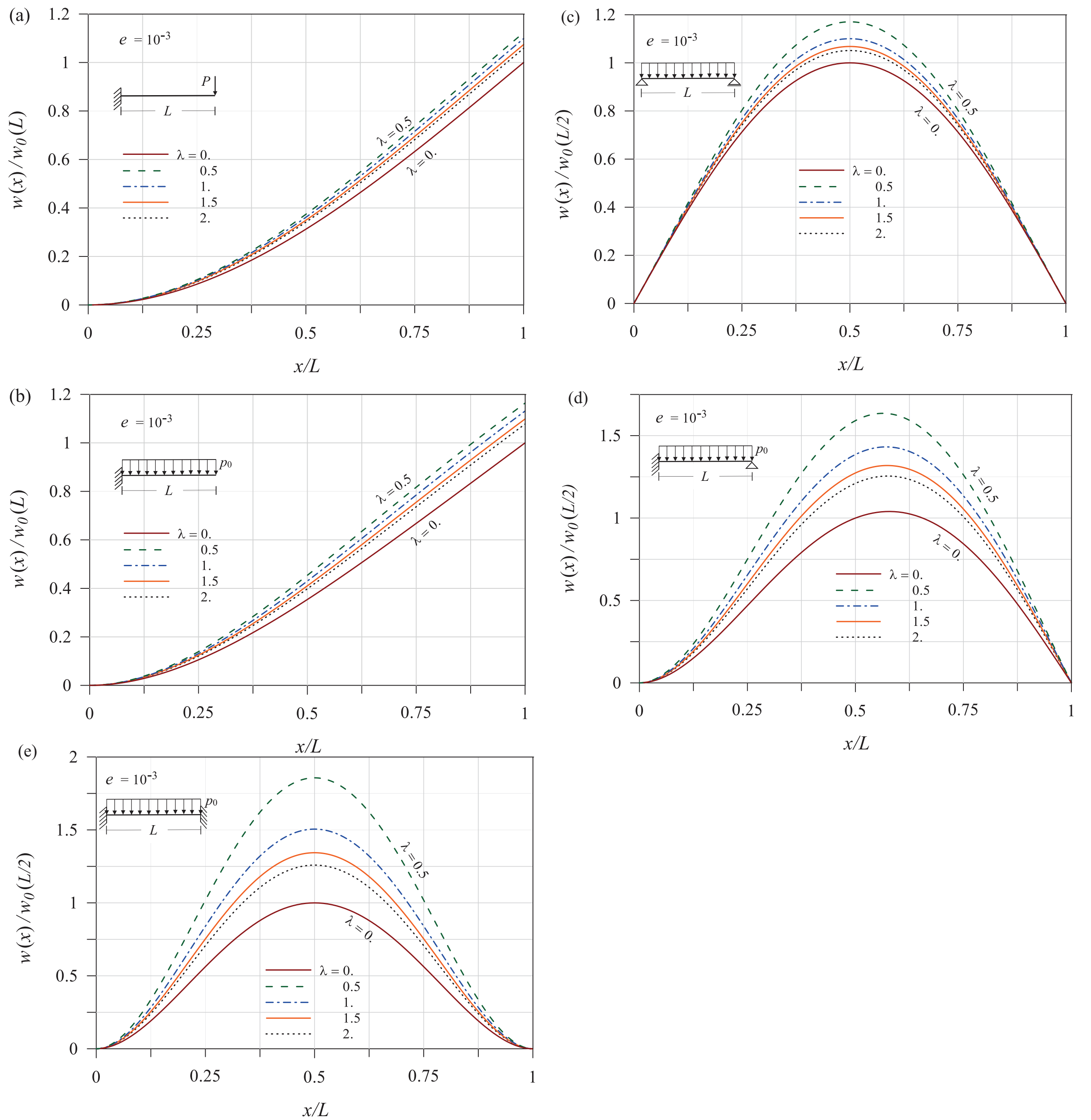

FIGURE 3 Normalized deflection $w(x)/w_0$ versus non-dimensional abscissa x/L for $e = 10^{-3}$ and $\lambda = 0., 0.5, 1., 1.5, 2.$: (a) CF1 case; (b) CF2 case; (c) PP case; (d) CP case; (e) CC case

since by size effects $\delta(\lambda) > 1$ for intermediate values of λ, but $\delta(0) = \delta(\infty) = 1$, necessarily the curve $\delta(\lambda)$ must possess a waved pattern.

7.2 | Solution procedure using the strain gradient elasticity theory

The strain gradient elasticity theory cast in a simplified form proposed by [74–77] leads to a sixth order differential equation as

$$\left[w(x) - \ell^2 w''(x)\right]'''' = p(x)/D \tag{78}$$

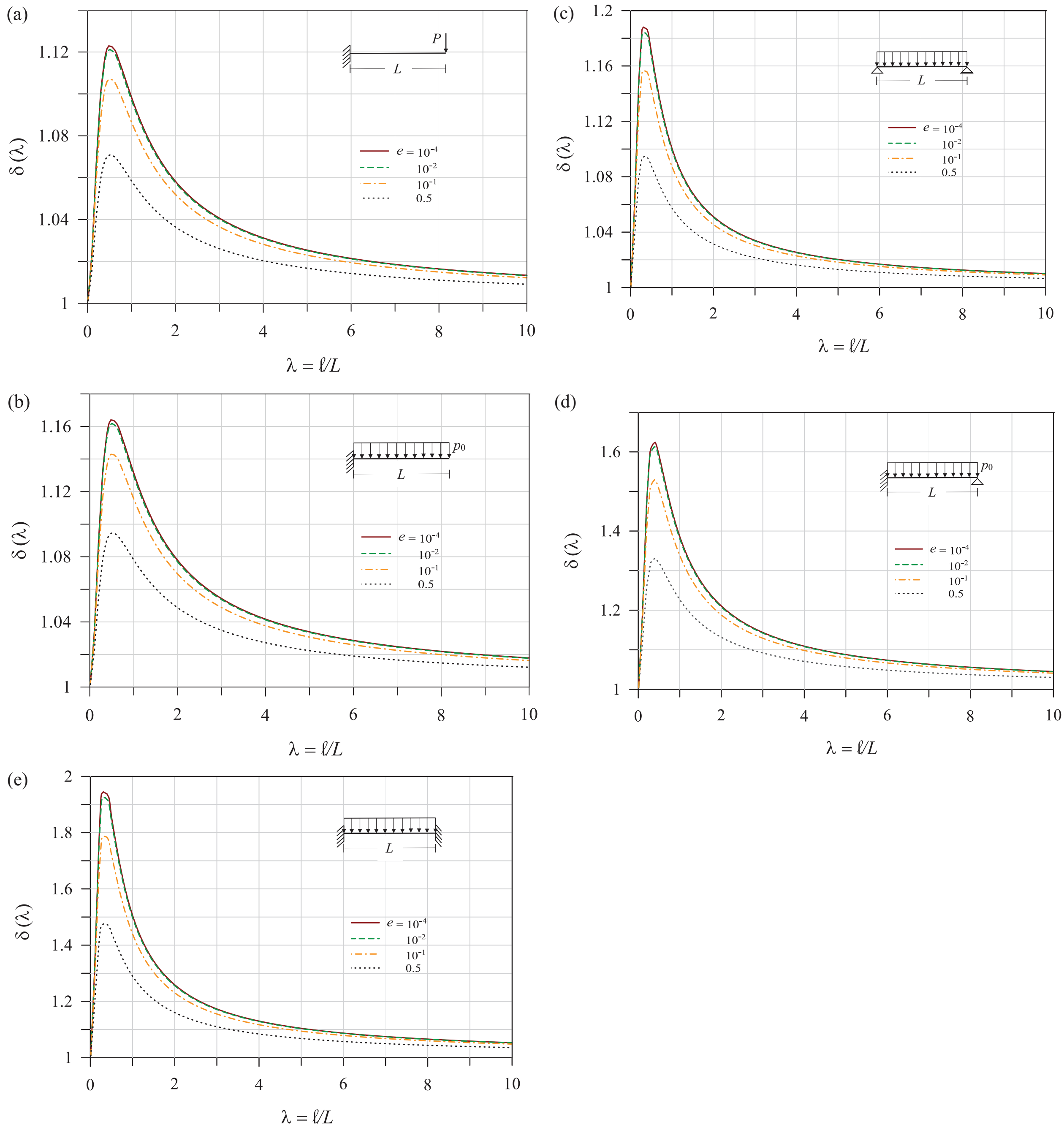

FIGURE 4 Normalized maximum deflection $\delta(\lambda)$ versus non-dimensional length scale parameter $\lambda = \ell/L$ for $e = 10^{-4}, 10^{-2}, 10^{-1}, 0.5$: (a) CF1 case; (b) CF2 case; (c) PP case; (d) CP case; (e) CC case

Though the length scale parameter has in general different values for gradient and nonlocal materials due to the different ways in which long distance actions are accounted for with the two constitutive models [41], for comparison purposes here we assume that ℓ has the same values for both models.

The general integral of (78) can be written as

$$w(x) = \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4 + C_5 \sinh\left(\frac{x}{\ell}\right) + C_6 \cosh\left(\frac{x}{\ell}\right) + R_1(x)/D \tag{79}$$

where $C_1, \ldots, C_6$ are arbitrary constants and $R_1(x)$ is a particular function satisfying the condition $R_1''''(x) \equiv p(x)$. There are six unknown constants which can be uniquely determined using, for every beam case, the related four ordinary boundary conditions, along with the two extra gradient boundary conditions. These require that either the curvature $\chi = -w''$, or the higher order bending moment $M^{(1)} = -\ell^2 D w'''$, must be assigned at the beam ends [76]. In the present applications we choose these extra boundary conditions in the form:

$$\left.\begin{array}{l} w''(0) = w'''(L) = 0 \text{ for CF1, CF2, CP cases} \\ w'''(0) = w'''(L) = 0 \quad \text{for PP case} \\ w''(0) = w''(L) = 0 \quad \text{for CC case} \end{array}\right\} \tag{80}$$

Namely, for convenience, our choice was that the curvature is fixed or free according to whether the rotation is fixed or free, but other choices may be adopted. Using the strain gradient theory the normalized maximum deflection $\delta(\lambda)$ has been computed as a function of λ and plotted in Figures 5(a-e) (dashed-dot lines). It is shown that strain gradient theory predicts stiffening size effects for all beam cases.

For comparison purposes the analogous curves pertaining to the present model (solid lines) are also reported in Figures 5(a-e). It is notable the different asymptotic behavior of the gradient model and the present one. Indeed, whereas the present model for $\lambda \to \infty$ tends to recover the local behavior, instead the gradient model tends correspondingly to behave as a perfectly rigid model ($\delta = 0$ for $\lambda \to \infty$). This latter behavior may be heuristically explained observing that in a gradient model even a small value of the strain gradients may induce, at larger values of λ, a notable increase of the (equilibrium) stresses.

7.3 | Solution procedure using Eringen's nonlocal/differential theory

The nonlocal elasticity theory in the differential form proposed by [2] and developed by [27] leads to a fourth order differential equation of the form

$$w''''(x) = \frac{1}{D}[p(x) - \ell^2 p''(x)] \tag{81}$$

in which the length scale parameter enter into play only through the distributed load. This differential equation requires only four boundary conditions like the classical one, *it thus is not allowed to be equivalent to the Eringen's fully nonlocal model*, for which purpose the two extra nonlocality boundary conditions (50) should be also accommodated. Therefore, the latter equation may also be qualified as the *non-equivalent* nonlocal differential equation.[6]

Nevertheless, (81) has been widely used for the analysis of size effects within micro/nano-technologies, it therefore would be useful to better understand the physical/mathematical meaning of this model by its own, independently of the integral model from which it has been derived. Indeed, (81) admits a *unique solution* of the form

$$w(x) = \frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 + C_3 x + C_4 + R_2(x)/D \tag{82}$$

where $C_1, .., C_4$ are unknown constants and $R_2(x)$ is a particular solution of the differential equation $R_2''''(x) = p(x) - \ell^2 p''(x)$. In the case of concentrated loads, in which $R_2(x)$ carries in singularities at the application points of the loads, solutions coinciding with the classical solutions are found as long as the solution is required to belong to the usual continuity framework; typical example is the CF1 beam case, known as the paradox case. The corresponding normalized maximum deflection $\delta(\lambda)$ curves, in most part provided by [27], are plotted in Figures 5(a-e) (dashed line plots). The anomalous scattering behavior of the latter dashed lines plots is evident on passing through the various beam cases. In particular in the CF2 case of Figure 5(b)—usually recalled for its predicted stiffening behavior with increasing λ—is another paradoxical case, which surprisingly has never been explicitly mentioned in the literature, to the authors' knowledge. Indeed, $\delta(\lambda)$ becomes negative for $\lambda > 0.5$, which means that at least a portion of the beam raises up against the applied (downward) load p_0, which makes the plot $\delta(\lambda)$ physically unacceptable for $\lambda > 0.5$ (dotted branch of the dashed line under consideration below the $\delta = 0$ axis).

[6] This property for (81) is true only for finite domains, not for unbounded ones. In the latter cases, in fact, the asymptotic evanescent values of the structural response at infinity make the extra nonlocality boundary conditions be automatically satisfied, hence (81) proves to be fully equivalent correspondingly.

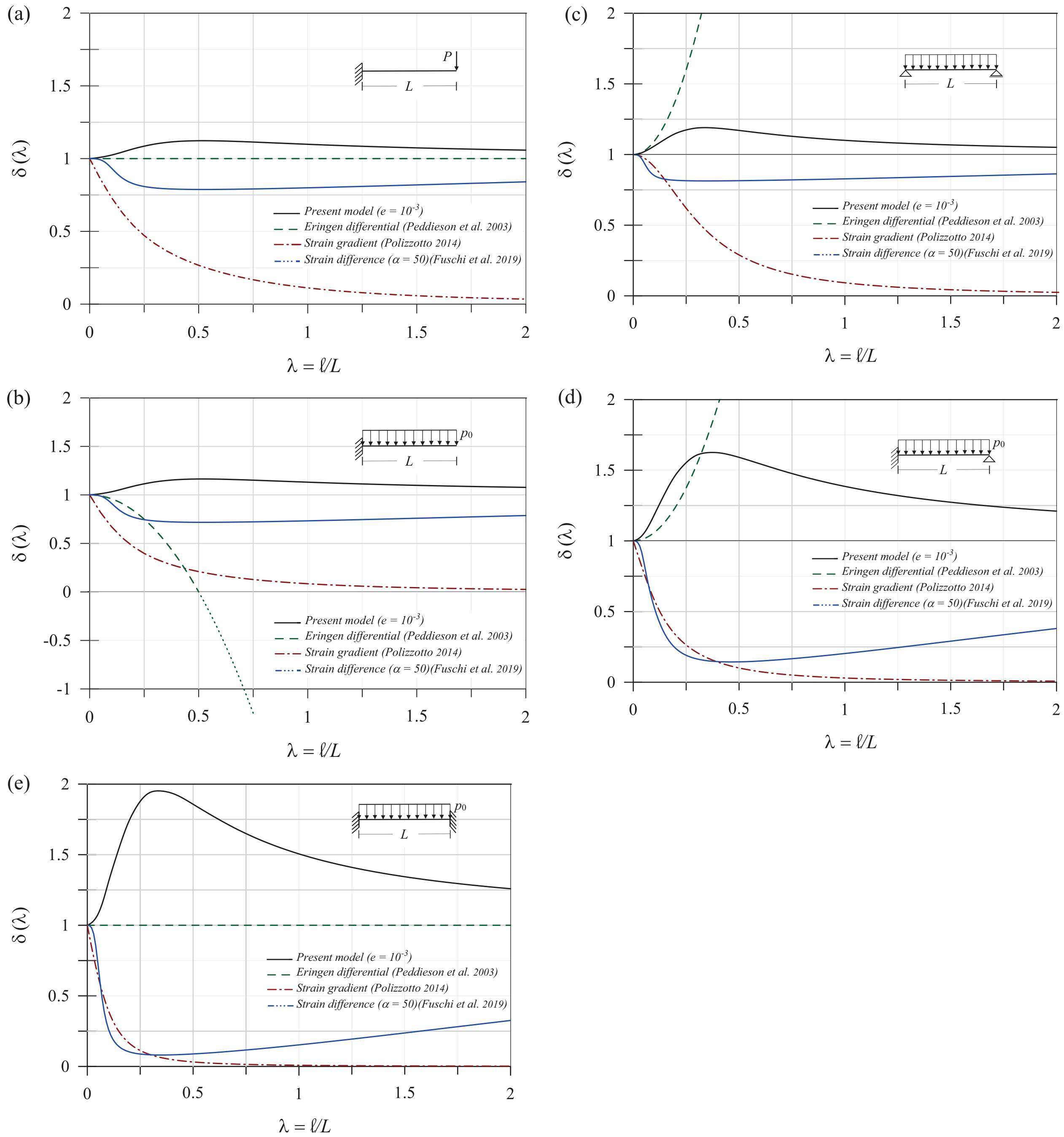

FIGURE 5 Normalized maximum deflection $\delta(\lambda)$ plotted as a function of non-dimensional length scale parameter $\lambda = \ell/L$ for the present model with $e = 10^{-3}$ (solid line), the Eringen differential model (dashed line), the strain gradient model (dash-dot line) and the strain difference model (dash-three dots line): (a) CF1 case; (b) CF2 case; (c) PP case; (d) CP case; (e) CC case. For CF2 case, the strain gradient model, the strain difference model and the Eringen differential one predict stiffening behavior, but the latter model predicts negative, hence physically unacceptable, values of $\delta(\lambda)$ for $\lambda > 0.5$ (dotted branch of the dashed line)

7.4 | Comparison with results by [68]

Koutsoumaris and co-workers [68] addressed the same beam problems discussed in the present paper using for this purpose the integral approach through Equation (31), but taking $e = 0$, hence $\xi(x) = 1 - \gamma_\ell(x)$. Equation (31) is addressed in the form as it is, in association with the four ordinary boundary conditions. Numerical solutions were reported for a discrete set of values of ℓ, that is, $\ell = (0.01, 0.015, 0.02, 0.025, 0.03)L$. The normalized values of the maximum deflection $\delta(\lambda)$ reported by [68] approximately agree with those obtained using the present method.

7.5 | Comparison with results by [69]

The authors of the present paper addressed in [69] the same beam problems discussed above by means of their strain-difference based nonlocal elasticity theory developed elsewhere [38]. The latter theory addresses a nonhomogeneous anisotropic elasticity theory which, in isothermal conditions, is thermodynamically featured by a Helmholtz free energy potential as

$$\psi = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\,\mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon}) : (\alpha\mathbf{C}) : \mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon}) \tag{83}$$

where $\mathbf{C} = \mathbf{C}(\mathbf{x})$ is the fourth order moduli tensor of anisotropic elasticity, α is a material coefficient with the role of phase parameter. Equation (83) is a quadratic form in the strain $\boldsymbol{\varepsilon}(\mathbf{x})$ and the nonlocal strain difference defined as

$$\mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon})(\mathbf{x}) := \int_V g_\ell(\mathbf{x},\bar{\mathbf{x}})[\underbrace{\boldsymbol{\varepsilon}(\bar{\mathbf{x}}) - \boldsymbol{\varepsilon}(\mathbf{x})}_{\mathcal{D}\boldsymbol{\varepsilon}(\mathbf{x},\bar{\mathbf{x}})}]\,\mathrm{d}V(\bar{x}) \tag{84}$$

in which the kernel function g_ℓ is of the form (2). For $\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}_0 =$ constant, it is $\mathcal{D}\boldsymbol{\varepsilon} \equiv \mathbf{0}$, hence the local stress recovery condition is automatically satisfied. The stress–strain relation is derived from (83) by writing [38]:

$$\left.\begin{aligned} \mathbf{t} &= \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = \mathbf{C} : \boldsymbol{\varepsilon} \\ \boldsymbol{\tau} &= \frac{\partial\psi}{\partial\mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon})} = \alpha\mathbf{C} : \mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon}) \\ \boldsymbol{\sigma} &= \mathbf{t} + \mathcal{R}(\mathcal{D}\boldsymbol{\tau}) \end{aligned}\right\} \tag{85}$$

On expanding Equation $(85)_3$ we obtain either

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) - \alpha\int_V \mathbf{J}(\mathbf{x},\bar{\mathbf{x}}) : [\boldsymbol{\varepsilon}(\bar{\mathbf{x}}) - \boldsymbol{\varepsilon}(\mathbf{x})]\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{86}$$

or, equivalently,

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) + \alpha\int_V \mathbf{S}(\mathbf{x},\bar{\mathbf{x}}) : \boldsymbol{\varepsilon}(\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) \tag{87}$$

where $\mathbf{J}$ and $\mathbf{S}$ denote alternative forms of the related nonlocal stiffness tensors, that is,

$$\mathbf{J}(\mathbf{x},\bar{\mathbf{x}}) := [\gamma_\ell(\mathbf{x})\mathbf{C}(\mathbf{x}) + \gamma_\ell(\bar{\mathbf{x}})\mathbf{C}(\bar{\mathbf{x}})]\,g_\ell(\mathbf{x},\bar{\mathbf{x}}) - \int_V g_\ell(\mathbf{x},\mathbf{z})g_\ell(\bar{\mathbf{x}},\mathbf{z})\mathbf{C}(\mathbf{z})\,\mathrm{d}V(\mathbf{z}) \tag{88}$$

and

$$\mathbf{S}(\mathbf{x},\bar{\mathbf{x}}) := \frac{1}{2}\left[\gamma_\ell^2(\mathbf{x})\mathbf{C}(\mathbf{x}) + \gamma_\ell^2(\bar{\mathbf{x}})\mathbf{C}(\bar{\mathbf{x}})\right]\delta_D(\mathbf{x},\bar{\mathbf{x}}) - \mathbf{J}(\mathbf{x},\bar{\mathbf{x}}) \tag{89}$$

satisfying the equalities

$$\left.\begin{aligned} &\int_V \mathbf{J}(\mathbf{x},\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) = \gamma_\ell^2(\mathbf{x})\mathbf{C}(\mathbf{x}) \\ &\int_V \mathbf{S}(\mathbf{x},\bar{\mathbf{x}})\,\mathrm{d}V(\bar{\mathbf{x}}) = \mathbf{0} \end{aligned}\right\}\quad \forall\mathbf{x}\in V \tag{90}$$

The constitutive equation (86), or (87), written in the appropriate form for EB beam models, was used in [69] in association with the standard equilibrium and compatibility equations to solve the benchmark beam problems discussed in the preceding. For this purpose, the theory under consideration was specialized to homogeneous isotropic materials and compact convex domains with a constant length scale parameter ℓ. Among other, the normalized maximum deflection function $\delta(\lambda)$ was computed for a series of benchmark EB beams; the obtained curves $\delta(\lambda)$ are reported in Figures 5(a-e) (dash-triple dot lines). It emerged that the strain-difference based nonlocal model predicts a stiffening behavior of the beam structures with increasing λ, and that the plot $\delta(\lambda)$ exhibits a waved pattern with a minimum somewhere within $0 < \lambda < 0.5$ and an asymptotic limit $\delta \to \delta_\infty \approx 1$, and this for all considered beam cases. The evident contrast/similarity features between the two nonlocal models can have a heuristic physical interpretation as follows (here homogeneous material is considered):

α) Whereas the strain-difference based nonlocal model is featured by the free energy potential ψ of Equation (83) incorporating a *quadratic* addend in the $\mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon})$ variable, the analogous potential of the present model has the form

$$\psi = (1+e)\frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C} : \mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon}) \tag{91}$$

which instead incorporates a *bilinear* term in $\boldsymbol{\varepsilon}$ and $\mathcal{R}(\mathcal{D}\boldsymbol{\varepsilon})$.

β) The latter bilinear term is obviously responsible for the softening behavior exhibited by the associated beam model (solid lines in Figures 5(a-e)), even for $e = 0$. Therefore, substituting the mentioned bilinear term with the quadratic one of (83), necessarily a stiffening behavior of the related beam model will be promoted (dash-triple dot lines in Figures 5(a-e)). In [72] the method presented in [69] is extended to shear-warping deformable beams.

7.6 | Further comments on the obtained results

As stated by [2, 41], a size dependent nonlocal beam model is expected to behave like an atomic lattice model when the length scale parameter becomes larger than unit, namely $\lambda \to \infty$. We found that the enhanced model confirms the expected limit behavior (which in fact predicts that $\delta(\lambda) \to \delta_\infty \approx 1$ as $\lambda \to \infty$). Indeed, in spite of the larger computational efforts required by the enhanced model with respect to other simpler models (like e.g. the two-phase local/nonlocal one) there are good motivations for a regularization of the Eringen's fully nonlocal model like the one advanced with the present enhanced model.

A comparison of the obtained results with experimental data would be paramount for the validation of the proposed theory. In the absence of such data (likely of difficult obtainment through laboratory work), this comparison has to remain lacking from the present paper, but there remain the hope that the offered theoretical framework may be in some way useful for future experimental applications. Simulation methods via electro-magneto-mechanical coupling applied to CNT and graphene sheet micro-devises—in which some electro-magnetic state field obeys a suitable nonlocal type, or nonlocal gradient type, constitutive law [48, 50, 78, 79]—may constitute a valid alternative to atomistic simulation and laboratory experimental methods.

In a recent paper [80] it has been experimentally demonstrated that materials may exhibit both stiffening, or softening, size effects depending on particular state conditions of the material. Indeed, according to [80], *smaller seems to be not always stiffer*. Therefore, none of the considered models seems to be able to cover the entire wide spectrum of material properties at small scales, but they may be combined with each other to generate a good nonlocal strain gradient theory [30, 41, 44, 45, 49]. But all this constitutes a promising ongoing research.

8 | CONCLUSIONS

Working within the framework of linear elasticity, infinitesimal displacements and static conditions, a novel theory of nonlocal Euler–Bernoulli beams has been presented, which is useful for the analysis of size effects at small scales. It constitutes an enhanced form of the Eringen's fully nonlocal model, to which a non-homogeneous local phase has been added with the regularizing role of removing boundary effects and ill-posedness of the related boundary-value problems. The proposed theory generally predicts softening size effects in a consistent manner, without paradoxes or other shortcomings, for all types of beams and boundary/load conditions. The response of the beam can be uniquely determined, either through an integral approach by solving Fredholm integral equations of the second kind, or equivalently through a differential approach by solving sixth order differential equations with related ordinary and extra nonlocality boundary conditions.

The results of numerous examples of engineering beam problems have been reported and graphically illustrated together with comparisons with analogous results obtained using other theories. The most meaningful response function therein used is the maximum deflection of the beam, $\delta(\lambda)$, varying with the length scale parameter λ. This function generally exhibits a waved pattern, with positive slopes for minor values of λ, negative for major values, and with an intermediate value $\lambda \approx 0.5$ at which δ takes on a maximum finite value; also, the enhanced beam possesses an asymptotic behavior for $\lambda \to \infty$ which seems to be like that of an atomic lattice model, in accord with [2, 41].

In concluding this paper, we can state that the main original contributions of the present research work can be summarized as follows:

a) Formulation of an enhanced integral approach to nonlocal Euler–Bernoulli beams by Fredholm integral equations of the second kind for the analysis of size effects without paradoxes or other drawbacks, which constitutes an improved form of the theory proposed by [68].
b) Formulation of the equivalent differential approach to the beam problem, including the exact form of the extra nonlocality boundary conditions, with which the same solution as the integral approach can be obtained.
c) Numerical procedure to solve every beam problem through the application of the superposition principle, whereby the beam problem is decomposed into three independent auxiliary beam problems, each of which is independent of the beam's boundary conditions.

We therefore may conclude asserting that the objectives announced at the beginning were satisfactorily achieved. The results herein provided are not at all exhaustive, further work is required, for instance to extend the proposed theory to the dynamic and buckling framework.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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