mathematics

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Existence of Nontrivial Solutions for Sixth-Order Differential Equations

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Abstract: We show the existence of at least one nontrivial solution for a nonlinear sixth-order ordinary differential equation is investigated. Our approach is based on critical point theory.

Keywords: sixth-order equations; critical points; existence

MSC: 34B15; 34B18; 35B38

Citation: Bonanno, G.; Candito, P.; O'Regan, D. Existence of Nontrivial Solutions for Sixth-Order Differential Equations. *Mathematics* **2021**, *9*, 1852. https://doi.org/10.3390/math9161852

Academic Editor: Alicia Cordero Barbero

Received: 22 July 2021
Accepted: 3 August 2021
Published: 5 August 2021

1. Introduction

In this paper, we investigate the existence of at least one nontrivial solution for the following nonlinear sixth-order boundary value problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(x,u), & x \in [a,b], \\ u(a) = u(b) = u''(a) = u''(b) = u^{(iv)}(a) = u^{(iv)}(b) = 0, \end{cases} \tag{1}$$

where $\lambda > 0$, A, B and C are constants and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function. Here and in the sequel, we assume that

$$(H): \quad \max\{-Ak,\ -Ak - Bk^2,\ -Ak - Bk^2 - Ck^3\} < 1,$$

where $k = \left(\frac{b-a}{\pi}\right)^2$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is an* L^1*-Carathéodory function, i.e.*

$(f)_1$ *$x \to f(x,s)$ is measurable for every $s \in \mathbb{R}$;*
$(f)_2$ *$s \to f(x,s)$ is continuous for almost every $x \in [a,b]$;*
$(f)_3$ *for all $\rho > 0$ the function $x \to \sup_{|s| \le \rho} f(x,s)$ belongs to $L^1([a,b])$.*

Our aim is to establish an existence result for problem (1) by using variational methods, i.e., looking for a solution as a critical point of the corresponding energy functional. Roughly speaking, the existence of at least one nontrivial solution is ensured whenever the nonlinear term $f(x,\cdot)$ has a uniform sublinear growth in a suitable bounded interval $[d,c]$ which could be far from zero and/or infinity. This allows us to detect an interval of parameters λ for which problem (1) admits at least one solution and, in addition, to establish the boundedness of solutions uniformly with respect to the parameter. Moreover, our main result (Theorem 3) shows that the existence of a solution for (1) is not strictly connected with the asymptotic behavior of the non linearity at zero and at infinity which is a key ingredient usually required to apply some classical topological and variational methods as, for instance, fixed point theorems [1,2] and critical point theorems [3]. In this paper, precisely, we exploit the variational framework developed in [4], where the existence of infinitely many solutions is proved under an oscillating behavior of the reaction term at zero or at infinity. Then, applying a non-zero local minimum theorem (see Theorem 2), we obtain the existence of a nontrivial solution by requiring a suitable behavior of growth of the nonlinear term only in a set possibly bounded (see (ii) of Theorem 3). Moreover, some consequences of the main result in the autonomous case are pointed out. In particular, it is highlighted that the key assumption assumes a simpler form (see (16) of Corollary 1) and we show that the sublinearity at zero of nonlinear term is enough to obtain a nontrivial solution (see Corollary 2). Finally, a concrete example of an application of Theorem 3, where the sublinearity at zero is not requested, is emphasized (see Example 1).

Here, as an example of an application of our main result, we present the following

Theorem 1. *Fix three non-negative constants A, B, C and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = +\infty.$$

Then, for each positive number b such that $b < \sqrt[6]{2\pi^4 / \int_0^1 g(s)ds}$, the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = g(u), \quad x \in [0,b], \\ u(0) = u(b) = u''(0) = u''(b) = u^{(iv)}(0) = u^{(iv)}(b) = 0, \end{cases} \tag{2}$$

admits at least one non-zero classical solution u such that $\|u\|_\infty \leq 1$.

It is worth noting that the above problem is independent of the parameters λ and it admits non-zero solutions for any continuous function g which is sublinear at 0, provided that the interval $[0, b]$ is small enough.

Sixth-order differential equations appear in the literature, for instance, in [5,6], where existence and multiplicity results are proved for (P_1) with a nonlinear term of polynomial type, by using a minimization theorem and Clark's theorem.

Finally, for completeness, we refer the reader interested to have an overview on the applications of high order differential equations to [7–9] and the references therein, and to [10,11], where non-local conditions are also considered.

2. Mathematical Background

Throughout the paper

$$X = \{u \in H^3(a,b) \cap H_0^1(a,b) : \ u''(a) = u''(b) = 0\}$$

denotes the real Banach space equipped with the norm

$$\|u\|_X = \left(\|u'''\|_2^2 + \|u''\|_2^2 + \|u'\|_2^2 + \|u\|_2^2\right)^{1/2} \quad \forall\, u \in X, \tag{3}$$

where $\|\cdot\|_2$ indicates the usual norm in $L^2(a,b)$ and $H^3(a,b)$, $H_0^1(a,b)$ are the classical Sobolev spaces. It is well known that $\|\cdot\|_X$ is induced by the inner product

$$\langle u, v\rangle = \int_a^b (u'''(x)v'''(x) + u''(x)v''(x) + u'(x)v'(x) + u(x)v(x))\, dx \quad \forall\, u,\ v \in X.$$

Clearly $(X, \|\cdot\|_X) \hookrightarrow (C^0(a,b), \|\cdot\|_\infty)$ and the embedding is compact. Moreover, arguing as in [4,6], we point out some useful Poincaré and Sobolev type inequalities.

Proposition 1. *For every $u \in X$, one has*

$$\|u^{(i)}\|_2 \leq \left(\frac{b-a}{\pi}\right)^{j-i} \|u^{(j)}\|_2, \quad i = 0,1,2,\ j = 1,2,3 \text{ with } i < j. \tag{4}$$

$$\|u\|_\infty \leq \frac{1}{2}\frac{(b-a)^{j-1/2}}{\pi^{j-1}}\|u^{(j)}\|_2, \quad j=1,2,3. \tag{5}$$

Proof. Bearing in mind the usual well known Poincaré inequality $\|u\|_2 \leq \left(\frac{b-a}{\pi}\right)\|u'\|_2$ for all $u \in H_0^1(a,b)$, see for instance [12], we also have $\|u''\|_2 \leq \left(\frac{b-a}{\pi}\right)\|u'''\|_2$ being $u'' \in H_0^1(a,b)$. Thus, it easy to see that (4) holds if we show that the case $i=1$, $j=2$ is true. The other possible combinations can be obtained by iterating the previous inequalities. To this end, by Hölder's inequality, we have

$$\|u'\|_2^2 \leq \|u\|_2\|u''\|_2 \leq \left(\frac{b-a}{\pi}\right)\|u'\|_2\|u''\|_2.$$

which ensures (4). While, (5) is a direct consequence of (4), bearing in mind that $\|u\|_\infty \leq \frac{(b-a)^{1/2}}{2}\|u'\|_2$ for all $u \in H_0^1(a,b)$, see [12]. □

Let us define the function $N : X \to \mathbb{R}$ by putting

$$N(u) = \|u'''\|_2^2 + A\|u''\|_2^2 + B\|u'\|_2^2 + C\|u\|_2^2, \qquad \forall u \in X.$$

From (4), adapting here the arguments developed in [4] to solve problem (1) when the interval is $[0,1]$, we can prove the following auxiliary results.

Proposition 2 (Proposition 2.2 in [4])**.** *Let* $k = \left(\frac{b-a}{\pi}\right)^2$. *The condition*

$$(H) \quad \max\{-Ak,\ -Ak-Bk^2,\ -Ak-Bk^2-Ck^3\} < 1,$$

holds if and only if one of the following is satisfied

$(H)_1$ $A \geq 0, B \geq 0, C \geq 0$;
$(H)_2$ $A \geq 0, B \geq 0, C < 0$ *and* $-Ak-Bk^2-Ck^3 < 1$;
$(H)_3$ $A \geq 0, B < 0, C \geq 0$ *and* $-Ak-Bk^2 < 1$;
$(H)_4$ $A \geq 0, B < 0, C < 0$ *and* $-Ak-Bk^2-Ck^3 < 1$;
$(H)_5$ $A < 0, B \geq 0, C \geq 0$ *and* $-Ak < 1$;
$(H)_6$ $A < 0, B \geq 0, C < 0$ *and* $\max\{-Ak,\ -Ak-Bk^2-Ck^3\} < 1$;
$(H)_7$ $A < 0, B < 0, C \geq 0$ *and* $-Ak-Bk^2 < 1$;
$(H)_8$ $A < 0, B < 0, C < 0$ *and* $-Ak-Bk^2-Ck^3 < 1$.

Moreover, setting

$$\delta = \begin{cases} 1 & \text{if } (H)_1 \text{ holds} \\ \min\{1, 1+Ak+Bk^2+Ck^3\} & \text{if } (H)_2 \text{ or } (H)_4 \text{ holds} \\ \min\{1, 1+Ak+Bk^2\} & \text{if } (H)_3 \text{ holds} \\ 1+Ak & \text{if } (H)_5 \text{ holds} \\ \min\{1+Ak, 1+Ak+Bk^2\} & \text{if } (H)_6 \text{ holds} \\ 1+Ak+Bk^2 & \text{if } (H)_7 \text{ holds} \\ 1+Ak+Bk^2+Ck^3 & \text{if } (H)_8 \text{ holds,} \end{cases} \tag{6}$$

we point out the following proposition.

Proposition 3. *Assume* (H). *Then, for every* $u \in X$, *the following conditions hold:*

$(N)_1$: $N(u) \geq \delta \|u'''\|_2^2$;
$(N)_2$: $N(u) \geq m\|u\|_X^2$, *with* $m := \delta/4$ *if* $b \leq a+\pi$ *or* $m := \frac{\delta}{4}\left(\frac{\pi}{b-a}\right)^6$ *if* $b > a+\pi$;
$(N)_3$: $N(u) \geq \dfrac{4\delta\pi^4}{(b-a)^5}\|u\|_\infty^2$.

Proof. The proof is similar to Propositions 2.3 and 2.5 in [4], so we give only an outline. For instance, assume that $(H)_1$ holds. Then, in view of (4) one has

$$\begin{aligned} N(u) &\geq \|u'''\|_2^2 \\ &\geq \frac{1}{4}\left(\|u'''\|_2^2 + \frac{1}{k}\|u''\|_2^2 + \frac{1}{k^2}\|u'\|_2^2 + \frac{1}{k^3}\|u\|_2^2\right) \\ &\geq \frac{1}{4}\min\left\{1, \frac{1}{k}, \frac{1}{k^2}, \frac{1}{k^3}\right\}\|u\|_X^2. \end{aligned}$$

Hence, $(N)_1$ and $(N)_2$ are satisfied and $(N)_3$ follows from (5) and $(N)_1$. □

To set the variational framework of problem (1), we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ as follows:

$$\Phi(u) = \frac{N(u)}{2}, \quad \Psi(u) = \int_a^b F(x, u(x))\, dx \quad \forall\, u \in X, \tag{7}$$

where $F(x,t) = \int_0^t f(x,s)\, ds$ for every $(x,t) \in [a,b] \times \mathbb{R}$.
Standard arguments show that Φ and Ψ are continuously Gâteaux differentiable, being in particular

$$\Phi'(u)(v) = \int_a^b \left(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\right) dx$$

and

$$\Psi'(u)(v) = \int_a^b f(x, u(x))v(x)\, dx$$

for every u, $v \in X$.

We recall that a weak solution of problem (1) is any $u \in X$ such that

$$\int_a^b \left(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\right) dx = \lambda \int_a^b f(x, u(x))v(x)\, dx, \tag{8}$$

for every $v \in X$. Hence, the weak solutions of (1) are exactly the critical points of the functional $\Phi - \lambda\Psi$. Moreover, arguing as Proposition 2.7 in [4] we get

Proposition 4. *Assume that* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, then every weak solutions of (1) is also a classical solution.*

Let X be a Banach space, to achieve our goal, the main tool used is a non trivial local minimum theorem, (see [13] and [Theorem 2.3] in [14]) for functionals of type $I_\lambda = \Phi - \lambda\Psi$, where Φ, $\Psi : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functions fulfilling a weak Palais–Smale condition, namely *for* $r \in \mathbb{R}$, $I_\lambda = \Phi - \lambda\Psi$ *is said to satisfy the* $(PS)^{[r]}$*-condition if any sequence* $\{u_n\}$ *such that*

(α_1) $\{I_\lambda(u_n)\}$ *is bounded,*
(α_2) $\|I'_\lambda(u_n)\|_{X^*} \to 0$ *as* $n \to \infty$,
(α_3) $\Phi(u_n) < r \quad \forall\, n \in \mathbb{N}$,

has a convergent subsequence. Finally, setting

$$\underline{\varphi}(r) := \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r}; \qquad \overline{\varphi}(r) := \sup_{u \in \Phi^{-1}(]0, r[)} \frac{\Psi(u)}{\Phi(u)},$$

we recall the non-zero local minimum theorem (see Theorem 2.3 of [13]).

Theorem 2. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exists $r > 0$ such that*

$$\underline{\varphi}(r) < \overline{\varphi}(r), \tag{9}$$

and for each $\lambda \in \Lambda_r := \left] \frac{1}{\overline{\varphi}(r)}, \frac{1}{\underline{\varphi}(r)} \right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition.

Then, for each $\lambda \in \Lambda_r$ there is $u_\lambda \in \Phi^{-1}(]0, r[)$ (hence, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u_\lambda \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_\lambda) = 0$.

3. Main Results

In this section, we present our main result and some of its consequences. To this end, put

$$K_1 := 96\left(\frac{12}{5}\right)^5 + 4\left(\frac{12}{5}\right)^4 A(b-a)^2 + \frac{1248}{175}B(b-a)^4 + \frac{493}{756}C(b-a)^6 \tag{10}$$

and

$$C_1 := \frac{4\delta\pi^4}{K_1}, \tag{11}$$

where δ is given in (6).

Our main result is the following.

Theorem 3. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that condition (H) holds and suppose that there exist two positive constants c and d with $d < c$ such that*

(i) *$F(x, s) \geq 0$ for all $(x, s) \in ([a, a + (5/12)(b-a)] \cup [b - (5/12)(b-a), b]) \times \mathbb{R}$,*

(ii) $\dfrac{\int_a^b \max_{|s|\leq c} F(x, s)\, dx}{c^2} < C_1 \dfrac{\int_{a+(5/12)(b-a)}^{b-(5/12)(b-a)} F(x, d)\, dx}{d^2}$.

Then, for each

$$\lambda \in \Lambda_{c,d} = \frac{2\delta\pi^4}{(b-a)^5}\left] \frac{1}{C_1} \frac{d^2}{\int_{a+(5/12)(b-a)}^{b-(5/12)(b-a)} F(x, d)\, dx}, \frac{c^2}{\int_a^b \max_{|s|\leq c} F(x, s)\, dx} \right[,$$

the problem (1) admits at least one non-zero weak solution $u_\lambda \in X$. Moreover, one has $\|u_\lambda\|_\infty < c$ and $\|u_\lambda\|_X < \left(\frac{4\pi^2}{(b-a)^{5/2}}(\max\{1, \frac{b-a}{\pi}\})^3\right) c$ for every $\lambda \in \Lambda_{c,d}$.

Proof. Our aim is to apply Theorem 2. To this end, we take $X = H^3(a, b) \cap H_0^1(a, b)$ and Φ and Ψ as in (7), which are, as recalled before, functionals of class C^1. Moreover, one has $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

Now, put

$$r := \frac{2\delta\pi^4}{(b-a)^5} c^2,$$

where δ is as in (6) and k as in Proposition 4. Moreover, if $\Phi(u) < r$ for some $u \in X$, taking into account $(N)_3$, it follows that

$$\|u\|_\infty < c.$$

Hence, it is easy to see that

$$\underline{\varphi}(r) := \frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} \leq \frac{(b-a)^5}{2\delta\pi^4} \frac{\int_a^b \max_{|s|\leq c} F(x, s)\, dx}{c^2} \tag{12}$$

Now, consider the functions:

$$T_a : \left[a, a + \frac{5}{12}(b-a)\right] \to [0,1], \quad T_b : \left[b - \frac{5}{12}(b-a), b\right] \to [0,1], \quad p, v : \mathbb{R} \to \mathbb{R}$$

defined by putting, respectively:

$$T_a(x) := \left(\frac{12}{5}\right)\frac{x-a}{b-a}, \qquad \forall\, x \in \left[a, a + \frac{5}{12}(b-a)\right],$$
$$T_b(x) := 1 - \left(\frac{12}{5}\right)\frac{x - [b - \frac{12}{5}(b-a)]}{b-a}, \qquad \forall\, x \in \left[b - \frac{5}{12}(b-a), b\right],$$
$$p(x) = x^4 - 2x^3 + 2x, \qquad \forall\, x \in \mathbb{R},$$

and

$$v(x) := \begin{cases} dp(T_a(x)) & \text{if } x \in [a, a + (5/12)(b-a)[, \\ d & \text{if } x \in [a + (5/12)(b-a), b - (5/12)(b-a)], \\ dp(T_b(x)) & \text{if } x \in]b - (5/12)(b-a), b]. \end{cases}$$

A direct computation shows that $v \in X$ with $\Phi(v) = \frac{1}{2}\frac{K_1}{(b-a)^5}d^2$. From $d < c$ it follows that $\Phi(v) < r$, that is $K_1 d^2 < 4\delta^2\pi^4 c^2$. Indeed, arguing by contradiction, we assume that $K_1 d^2 \geq 4\delta^2\pi^4 c^2$, then, since $d < c$, one has

$$\frac{\int_a^b \max_{|s|\leq c} F(x,s)\, dx}{c^2} \geq \frac{4\delta^2\pi^4}{K_1}\frac{\int_{a+(5/12)(b-a)}^{b-(5/12)(b-a)} F(x,d)\, dx}{d^2},$$

which contradicts assumption (ii) and our claim is proved.

Therefore, the previous computations and hypothesis (i) ensure that

$$\overline{\varphi}(r) := \sup_{u \in \Phi^{-1}(]0,r[)} \frac{\Psi(u)}{\Phi(u)} \geq \frac{\Psi(v)}{\Phi(v)} \geq \frac{2(b-a)^5}{K_1}\frac{\int_{a+(5/12)(b-a)}^{b-(5/12)(b-a)} F(x,d)\, dx}{d^2}. \tag{13}$$

Finally, from (12), (13) and (ii) we obtain (9) and $\Lambda_{c,d} \subseteq \Lambda_r$.

Now, in order to complete the proof, we are going to verify that the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition. To this end, let $\{u_n\}$ be a sequence in X such that $\|I'_\lambda(u_n)\|_{X^*} \to 0$ as $n \to \infty$, and $\Phi(u_n) < r \quad \forall\, n \in \mathbb{N}$. By $(N)_2$ and $(N)_3$, it follows that

$$\{u_n\} \text{ is bounded in } X \quad \text{and} \quad \|u_n\|_\infty \leq c \quad \forall\, n \in \mathbb{N}, \tag{14}$$

respectively. Furthermore, since X is a reflexive Banach space compactly embedded in $C^2([a,b])$, arguing by subsequences if necessary, one has that there exists $u \in X$ such that $u_n \rightharpoonup u$ in X and ,

$$\|u_n - u\|_\infty \to 0, \qquad \|u'_n - u'\|_\infty \to 0, \qquad \|u''_n - u''\|_\infty \to 0. \tag{15}$$

Now, since f is an L^1-Carathéodory function and (15) holds, it follows that there exists $\eta = \eta(c) \in L^1([a,b])$ such that

$$\lim_{n\to+\infty} |\Psi'(u_n)(u_n - u)| \leq \|\eta\|_1 \lim_{n\to+\infty} \|u_n - u\|_\infty \to 0.$$

Moreover, taking (14) into account, since $I'_\lambda(u_n)(u_n - u) \leq \|I'_\lambda(u_n)\|_{X^*}\|u_n - u\|_X \leq C\|I'_\lambda(u_n)\|_{X^*}$, one has

$$\lim_{n\to+\infty} I'_\lambda(u_n)(u_n - u) = 0,$$

for which it follows that

$$\lim_{n\to+\infty} \Phi'(u_n)(u_n - u) = \lim_{n\to+\infty} I'_\lambda(u_n)(u_n - u) + \lambda \lim_{n\to+\infty} \Psi'(u_n)(u_n - u) = 0.$$

Therefore, since $\Phi'(u) \in X^*$, one has

$$\lim_{n\to+\infty} \Phi'(u_n)(u_n - u) - \Phi'(u)(u_n - u) = 0,$$

and, then, follows

$$\lim_{n\to+\infty} N(u_n - u) = 0.$$

Hence, from $(N)_2$ one has that $\{u_n\}$ is strongly converging to $u \in X$ and our claim is proved.

Hence, since all the assumptions of Theorem 2 are satisfied the functional I_λ admits at least one non-zero critical point $u_\lambda \in X$, for each $\lambda \in \Lambda_{c,d}$, that is, u_λ is a non-zero solution of problem (1). Finally, since u_λ belongs to $\Phi^{-1}(]0,r[)$, from Proposition (3) the conclusion is achieved. □

Remark 1. According to Theorem 2 the non-zero solution u_λ of problem (1) ensured by Theorem 3 is a local minimum for the energy functional I_λ. Precisely, one has

$$I_\lambda(u_\lambda) = \min_{\|u\|_X<r} I_\lambda(u),$$

where $r = \left(\frac{4\pi^2}{(b-a)^{5/2}}(\max\{1, \frac{b-a}{\pi}\})^3\right)c$, with c as in the assumption (ii).

Remark 2. When $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous, Proposition 4 ensures that the solution guaranteed by Theorem 3 is a classical solution for problem (1).

Now, we point out some consequences of Theorem 3. To this end, put

$$\sigma = \sigma(a,b,\delta,\alpha) = \frac{4\delta\pi^4}{K_1}\left(\frac{\int_{a+(5/12)(b-a)}^{b-(5/12)(b-a)} \alpha(x)dx}{\int_a^b \alpha(x)dx}\right) = C_1 \frac{\int_{a+(5/12)(b-a)}^{b-(5/12)(b-a)} \alpha(x)dx}{\|\alpha\|_1},$$

where $\alpha \in L^1([a,b])$ *is a non negative and non-zero function.*

A first special case of Theorem 3 is the following.

Corollary 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and put $G(s) := \int_a^s g(t)\,dt$ for all $s \in \mathbb{R}$. Assume that condition (H) holds and suppose that there exist two positive constants d and c, with $d < c$, such that*

$$\frac{G(c)}{c^2} < \sigma \frac{G(d)}{d^2}. \tag{16}$$

Then, for each

$$\lambda \in \Lambda'_{c,d} = \frac{2\delta\pi^4}{(b-a)^5\|\alpha\|_1}\left]\frac{1}{\sigma}\frac{d^2}{G(d)}, \frac{c^2}{G(c)}\right[,$$

the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda\alpha(x)g(u), & x \in [a,b], \\ u(a) = u(b) = u''(a) = u''(b) = u^{(iv)}(a) = u^{(iv)}(b) = 0, \end{cases} \tag{17}$$

admits at least one non-zero weak solution $u_\lambda \in X$. Moreover, one has $\|u_\lambda\|_\infty < c$ and $\|u_\lambda\|_X < \left(\frac{4\pi^2}{(b-a)^{5/2}}(\max\{1, \frac{b-a}{\pi}\})^3\right)c$ for every $\lambda \in \Lambda'_{c,d}$.

Proof. It is enough to apply Theorem 3 with $f(x,s) = \alpha(x)g(s)$ for all $(x,s) \in [a,b] \times \mathbb{R}$. □

Remark 3. We observe that condition (G_σ) of Corollary 1 is satisfied, for instance, whenever

(G^*) $\liminf_{c\to+\infty} \frac{G(c)}{c^2} < \sigma \limsup_{d\to 0^+} \frac{G(d)}{d^2}$,

with

$$\Lambda'_{c,d} = \frac{2\delta\pi^4}{(b-a)^5\|\alpha\|_1}\left]\frac{1}{\sigma}\liminf_{d\to+0}\frac{d^2}{G(d)}, \limsup_{c\to+\infty}\frac{c^2}{G(c)}\right[.$$

Remark 4. We emphasize that, when g is non-negative, further suitable conditions on A, B, C ensure that the solution obtained by Theorem 3 is positive (see Remark 3.4 in [4]). For instance, if $A = B = 3$ and $C = 1$ (see Example 3.5 in [4]), Corollary 1 ensures the existence of at least one positive solution.

We point out another consequence of Theorem 3, where no sign condition on g is assumed.

Corollary 2. *Assume condition* (H) *and let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

(G_{0^+}): $\limsup_{s\to 0^+} \dfrac{G(s)}{s^2} = +\infty.$

Then, for each

$$\lambda \in \Lambda_c = \left]0, \frac{2\delta\pi^4}{(b-a)^5\|\alpha\|_1} \sup_{c>0} \frac{c^2}{\max_{|s|\le c} G(s)}\right[,$$

problem (17) admits at least one non-zero weak solution $u_\lambda \in X$.

Proof. Fix $\lambda \in \Lambda_c$. Therefore, there exist $c_\lambda > 0$ such that

$$\frac{1}{\lambda} > \frac{(b-a)^5\|\alpha\|_1}{2\delta\pi^4} \frac{\max_{|s|\le c_\lambda} G(s)}{c_\lambda^2}.$$

From (G_{0^+}), there is $d_\lambda > 0$, with $d_\lambda < c_\lambda$, such that

$$\frac{(b-a)^5\|\alpha\|_1}{2\delta\pi^4}\sigma \frac{G(d_\lambda)}{d_\lambda^2} > \frac{1}{\lambda}.$$

Hence, arguing as in Corollary 1, Theorem 3 ensures the conclusion. □

Remark 5. Theorem 1 in the Introduction is a further consequence of Theorem 3 obtained arguing as in the proof of Corollary 2 and by choosing $c = 1$.

Finally, we give an example where no condition either at infinity or zero is requested.

Example 1. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$g(s) = \begin{cases} s, & \text{if } s \le 1, \\ \frac{1}{s^3} & \text{if } 1 < s < 100, \\ h(s) & \text{if } s \ge 100, \end{cases}$$

where $h : [100, +\infty[\to \mathbb{R}$ is a completely arbitrary function. Without loss of generality, we can consider h continuous in $[100, +\infty[$ and such that $h(100) = \dfrac{1}{100^3}$.
Owing to Corollary 1, for each $\lambda \in]24, 78[$, the problem

$$\begin{cases} -u^{(vi)} - u^{(iv)} = \lambda x^2 g(u), & x \in [0,3], \\ u(0) = u(3) = u''(0) = u''(3) = u^{(iv)}(0) = u^{(iv)}(3) = 0, \end{cases}$$

admits at least one non-zero classical solution u_λ such that $\|u_\lambda\|_\infty < 100$ and $\|u_\lambda\|_X < \frac{400}{9\sqrt{3}}\pi^2$. Indeed, in this case, we have $a = 0$, $b = 3$, $\alpha(x) = x^2$, $x \in [0,3]$, $A = -1$, $B = C = 0$ for which a simple computation shows that $\sigma > 6 \times 10^{-4}$ and

$$G(s) = \begin{cases} \frac{s^2}{2}, & \text{if } s \le 1, \\ 1 - \frac{1}{2s^2} & \text{if } 1 < s < 100, \\ 1 - \frac{1}{2}\frac{1}{10^4} + \int_{100}^{s} h(t)dt & \text{if } s \ge 100, \end{cases}$$

so that by picking $d = 1$ and $c = 10^2$ the condition (16) is verified, $]24, 78[\subseteq \Lambda'_{c,d}$ and our claim is proved.

Author Contributions: Investigation, G.B., P.C. and D.O. All authors have read and agreed to the published version of the manuscript.

Funding: The paper is partially supported by PRIN 2017—Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW).

Acknowledgments: The first two authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflicts of Interest: The authors declare no conflict of interest.

References and Note

1. Li, W. The existence and multiplicity of positive solutions of nonlinear sixth-order boundary value problem with three variable coefficients. *Bound. Value Probl.* **2012**, *2012*, 1–16. [CrossRef]
2. Zhang, L.; An, Y. Existence and multiplicity of positive solutions of a boundary-value problem for sixth-order ODE with three parameters. *Bound. Value Probl.* **2010**, *2010*, 1–13. [CrossRef]
3. Gyulov, T.; Morosanu, G.; Tersian, S. Existence for a semilinear sixth-order ODE. *J. Math. Anal. Appl.* **2006**, *321*, 86–98. [CrossRef]
4. Bonanno, G.; Livrea, R. A sequence of positive solutions for sixth-order ordinary nonlinear differential problems. *Electron. J. Qual. Theory Differ. Equ.* **2021**, *17*, 1–17. [CrossRef]
5. Chaparova, J.; Peletier, L.A.; Tersian, S. Existence and nonexistence of nontrivial solutions of semilinear fourth- and sixth-order differential equations. *Adv. Differ. Equ.* **2003**, *8*, 1237–1258.
6. Tersian, S.; Chaparova, J. Periodic and homoclinic solutions of some semilinear sixth-order differential equations. *J. Math. Anal. Appl.* **2002**, *272*, 223–239. [CrossRef]
7. Bates, P.; Fife, P.; Gardner, R.; Jones, C.K.R.J. The existence of traveling wave solutions of a generalized phase-field model. *SIAM J. Math. Anal.* **1997**, *28*, 142–149. [CrossRef]
8. Caginalp, G.; Fife, P. Higher order phase field models and detailed anisotropy. *Phys. Rev. B* **1986**, *34*, 4940–4943. [CrossRef] [PubMed]
9. Gardner, R.A.; Jones, C.K.R.T. Traveling waves of a perturbed diffusion equation arising in a phase field model. *Indiana Univ. Math. J.* **1990** *39*, 1197–1222. [CrossRef]
10. Webb, J.R.L. Positive solutions of some higher order nonlocal boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **2009**, *29*, 15. [CrossRef]
11. Webb, J.R.L.; Zima, M. Multiple positive solutions of resonant and non-resonant non-local fourth-order boundary value problems. *Glasg. Math. J.* **2012**, *54*, 225–240. [CrossRef]
12. Amoroso, E.; Bonanno, G.; D'Aguì, G.; De Caro, S.; Foti, S.; O'Regan, D.; Testa, A. Second Order Differential Equations for the Power Converters Dynamical Performance Analysis, Preprint.
13. Bonanno, G. A critical point theorem via the Ekeland variational principle. *Nonlinear Anal.* **2012**, *75*, 2992–3007. [CrossRef]
14. Bonanno, G. Relations between the mountain pass theorem and local minima. *Adv. Nonlinear Anal.* **2012**, *1*, 205–220. [CrossRef]