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On the uniqueness of the solution for a semi-linear elliptic boundary value problem of the membrane MEMS device for reconstructing the membrane profile in absence of ghost solutions

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ABSTRACT

In this paper, the authors present a new condition of the uniqueness of the solution for a previous $1D$ semi-linear elliptic boundary value problem of membrane MEMS devices, where the amplitude of the electric field is considered proportional to the curvature of the membrane. The existence of the solution (membrane deflection) depends on the material of the membrane, which is obtained by Schauder–Tychonoff's fixed point approach. Thus, in this paper, the result of uniqueness has been completely reformulated to obtain a condition depending on the material of the membrane achieving a new result of existence and uniqueness, depending on both the material of the membrane and the geometrical characteristics of the device. Then, by shooting numerical method, more realistic conditions for detecting eventual ghost solutions and new ranges of both operational parameters and mechanical tension of the membrane ensuring convergence have been achieved confirming the useful information on the industrial applicability of the model under study.

1. Introduction

Nowadays, the new industrial guidelines oblige researchers and designers to develop low-cost sensors and actuators that can combine the physical nature of the problem under study and low-level machine languages. In this domain, for example, micro-electro-mechanical-systems (MEMSs) (both static and dynamic) represent some of the most important innovations of micro engineering because numerical modeling can model situations that may be overlapped with industrial reality [1,2]. However, the main problem is that modeling often does not permit the obtainment of explicit analytical solutions, for which we must be satisfied with conditions that ensure existence and uniqueness (without prejudice to the fact that the problem could be dealt with from a numerical viewpoint but taking precautions from any ghost solutions [3–5]). In the scientific literature, there are various study fronts that range from thermoelastic systems [6,7] to biomedical applications [8]. From the theoretical viewpoint, many MEMS models (sophisticated with strong non-linearity) have been elaborated to obtain the conditions of existence, uniqueness and regularity of the solution (in some cases) [9–11]. However, these models provide conditions where the properties of the materials of the device are not explicitly involved; thus, they are difficult to implement and not very interesting from an industrial viewpoint. One of the most accredited models considers a dimensionless MEMS device that consists of two metal plates: one is fixed, and the other is deformable but anchored to the edges; the applied electric voltage moves the deformable plate towards the fixed one. Its differential model in general terms, in which u represents the deflection of the deformable plate, is:

$$\begin{cases} \alpha_1 \Delta^2 u = \left(\beta \int_\Omega |\nabla u|^2 dx + \gamma\right)\Delta u + \dfrac{\lambda_1 f_1(x)}{(1-u)^\sigma \left(1+\chi \int_\Omega \frac{dx}{(1-u)^{\sigma_1 - 1}}\right)} \\ u = \Delta u - d u_{\alpha_1} = 0, \quad x \in \partial\Omega, \quad d \geq 0 \\ 0 < u < 1, \quad x \in \Omega \end{cases} \tag{1}$$

where the dielectric characteristics of the material are represented by the bounded function f_1; the applied voltage is λ_1; and α_1, β, γ and χ are physical parameters related to the mechanic and electric characteristics of the material (for a better understanding of the symbols present in the text, see Table 1). Finally, to account for more general electrostatic potentials, the exponent $\sigma_1 \geq 2$ is considered. However, this model considers the deformable plate with a given thickness. Therefore, if we want to model membrane devices, we must neglect both plate thickness and inertial effects. In other words, by imposing $\sigma_1 = 2$, $\alpha_1 = 1$, $\beta = 0$,

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$\gamma = 0$ and $\chi = 0$ (1) is simplified as follows [12]:

$$\begin{cases} \Delta^2 u(x) = \frac{\lambda_1 f_1(x)}{[1-u(x)]^2} \\ 0 < u(x) < 1 \quad \text{in } \Omega, \\ u = \Delta u - d u_{\alpha_1}, \quad \text{on } \partial\Omega, \quad d \geq 0 \end{cases} \tag{2}$$

In this work, starting from (2), where the bottom plate is replaced by a thin membrane attached to the edge, we obtain the following boundary elliptical semi-linear model:

$$\begin{cases} u'' = -f_2(x)\frac{\lambda_1}{(1-u(x))^2} & \text{in } \Omega = [-L, L] \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

For the $1D$ model in [13], where the applied voltage V (linked to parameter λ_1) is expressed in terms of the amplitude of the electrostatic field $|\mathbf{E}|$, $\mathbf{E}$ on the membrane is a locally orthogonal vector to the tangent direction of the membrane. Thus, $|\mathbf{E}|$ is considered proportional to the curvature K of the membrane to build a model where singularity $1-u(x)$ does not explicitly appear. The main results in [13] concern the conditions that ensure the existence and uniqueness of the solution to the formulated $1D$ problem. Although the condition of existence strongly depends on the electromechanical characteristics of the material of the membrane, the condition that ensures the uniqueness does not present the same peculiarities. Then, this work aims to obtain, firstly, a new condition that ensures the uniqueness of the solution depending on both the electromechanical parameters of the material constituting the membrane and the applied voltage in order to achieve a new condition of existence and uniqueness of the solution for the problem under study with the same peculiarities. In addition, starting from this new condition, the authors, by a numerical approach based on shooting procedure, reconstruct the profile of the membrane highlighting new range of values of electromechanical characteristics of the membrane (mechanical tension of the membrane, σ) and operational parameters (pair applied voltage and sup of $|\mathbf{E}|$, $(V, sup|\mathbf{E}|)$) that, being achieved from the new condition of existence and uniqueness,[1] are more adherent to reality and, then, more interesting from the industrial point of view. The remainder of the paper is structured as follows. Section 2 recalls the $1D$ model studied in [13], where $|\mathbf{E}|$ is considered proportional to K, $|\mathbf{E}(x)| = \mu(x, u(x), V)K(x, u(x))$ with V is the voltage applied, which formulates the problem in the Dirichlet form considering the safety distance d^* (i.e., the distance to the top of the membrane profile from the upper plate). Then, Section 3 shows the most important results concerning the existence of the solution of the problem with sketches of the proofs while . The core of this paper is Section 4, where the authors present a new result of the uniqueness of the solution, which depends on the electromechanical properties of the membrane for industrial applications. Unlike previous works (see [13]), where the study of the uniqueness of the solution did not impose any conditions, here, we must obtain a condition that ensures both existence and uniqueness, as shown in Section 5. In Section 6, after to have solved the problem numerically by means of shooting procedure and reconstruct the profile of the membrane, some results of convergence regarding the new ranges of σ and $(V, sup|\mathbf{E}|)$ have been carried out. Finally, some numerical considerations complete the study particularly in reference to the validity of the numerical methodologies in [3], which are also applicable with the conditions of existence and uniqueness here. Finally, some remarks conclude the work in Section 7.

2. An overview on the formulation for $|\mathbf{E}|$ in terms of curvature of the membrane

2.1. Some physical backgrounds

To understand how a membrane MEMS works, let us consider a system of Cartesian axes $O'x'y'z'$ in $\mathbb{R}^3$, where an electrostatic–elastic

Table 1
List of the useful symbols.

Symbol	Description
$\mathbf{E}$	Electrostatic field
ϵ_0	Permittivity of the free space
$2L$	Length of the device
$2L_1$	Length of the device (dimensionless)
σ	Mechanical tension of the membrane
h	Height of the device
$\alpha_1, \beta, \gamma, \chi$	Electromechanical parameters of the material constituting the membrane
σ_1	Coulomb exponent
V	Applied electric voltage
ϕ	Electrostatic potential
Φ	Electrostatic potential (dimensionless)
D	Flexure rigidity of the membrane
d^*	Critical distance
u	Deflection of the membrane (dimensionless)
ϵ_t	Dielectric strength of the membrane
α	$1 - d^*$
M	$sup\{\lambda\}$
H	$sup\{\lvert u'\rvert\}$
$\theta \in \mathbb{R}^+$	Coefficient of proportionality between $-u''$ and $\lvert\mathbf{E}\rvert^2$
$f_1(x)$	Electrical properties of the membrane (bounded function)
K	Curvature of the membrane
$\delta = D/((2L)^2\sigma)$	Relative importance of tension and rigidity
$G(x, s)$	Green function
$\overline{H}$	$sup\lvert H\rvert$
$\overline{\lambda}^2 > 0$	Minimum voltage to apply to the device to win the mechanical inertia of the membrane

MEMS occurs because of a pair of parallel plates (located normally to the axis z') with a length of $2L$ and a mutual distance h; one plate is fixed, and the other is elastic (and deformable) but fixed to the edges. An electrostatic voltage V is applied to the device, whose fixed plate is at zero potential ($\phi = 0$), so that $\Delta\phi = 0$ in the zone bounded by the plates [1,13]. Considering as dependent variable the elastic deflection of the elastic plate, w_1, and indicating by σ the mechanical tension in the plate, D is the flexural rigidity, and ϵ_o is the permittivity of free space, using suitable scaling factors, we obtain the following system of nonlinear coupled partial differential equations [1,13]:

$$\begin{cases} \epsilon^2 \Delta_\perp \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \\ -\Delta_\perp w + \delta \Delta_\perp^2 w = -\lambda^2 \left[\epsilon^2 |\nabla_\perp \Phi|^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] \\ \Phi = 1 \quad \text{on elastic plate} \\ \Phi = 0 \quad \text{on fixed plate.} \end{cases} \tag{4}$$

In (4) Φ and w are the scaled electrostatic potential and the scaled deflection of the elastic plate (that is in dimensionless conditions) while

$$\delta = D/((2L)^2\sigma); \quad \epsilon = h/(2L) \tag{5}$$

and, in addition,

$$\lambda_1 = \lambda^2 = \frac{\epsilon_0 V^2 (2L)^2}{2h^3\sigma} = \beta V^2 \tag{6}$$

is the ratio of a reference electrostatic force to a reference elastic force, and

$$\beta = \frac{\epsilon_0 (2L)^2}{2h^3\sigma} \tag{7}$$

considers the electro-mechanical characteristics of the material of the membrane that, in dimensionless conditions, becomes

$$\beta_1 = \epsilon_0/(2\sigma) \geq 10^{12}. \tag{8}$$

However, experimentally, the values of ϱ are greater than 10^{12}. Because modern technologies enable us the extreme exploitation of materials, D can be neglected. Thus, by neglecting the thickness and width of the device with respect to its length ($1D$ model) and replacing

[1] Depending on both characteristics of the membrane and operational parameters

the deformable plate with a membrane anchored to the fixed plated, which serves only as a membrane support, in stationary conditions, we can write:

$$\begin{cases} u'' = -\frac{\lambda^2}{(1-u)^2} \quad \text{in } \Omega = [-L_1, L_1] \\ u(-L_1) = u(L_1) = 0 \end{cases} \tag{9}$$

where axes z is reversed, $L_1 = 0.5$ represents dimensionless L and u is the new deflection of the membrane.

2.2. Membrane MEMS: A well-known 1D model

As studied in [13], in (9), considering (6), λ^2 is proportional to V^2, so $\lambda^2(1-u)^{-2} \propto |\mathbf{E}|^2$. Because θ indicates this function of proportionality, (9) assumes the following form:

$$\begin{cases} -u'' = \theta|\mathbf{E}|^2 \quad \text{in} \quad \Omega = [-L_1, L_1] \\ u(-L_1) = u(L_1) = 0 \quad \theta \in \mathbb{R}. \end{cases} \tag{10}$$

In [13], since the line of force of $\mathbf{E}$ is orthogonal to the tangent of the membrane surface at each point on the membrane, $|\mathbf{E}|$ is expressed as the product of the membrane curvature K and function of proportionality μ:

$$|\mathbf{E}(x)| = \mu(x, u(x), \lambda) K(x, u(x)) \tag{11}$$

where

$$\begin{cases} \mu(x, u(x), \lambda) = \frac{\lambda}{(1-u(x)-d^*)} \\ \mu(x, u(x), \lambda) \in C^0([-L_1, L_1] \times [0, 1) \times [\bar{\lambda}, M]) \end{cases} \tag{12}$$

where $\bar{\lambda}^2$ and $M = sup\{\lambda\}$ are, less than coefficients that regulate any proportionality, the minimum voltage for the device to win the inertia of the membrane and the maximum admissible voltage, respectively; $d^* = \lambda(\epsilon_t)^{-1}$; ϵ_t is the dielectric strength of the material of the membrane, which represents the critical distance to ensure that even if u reaches its maximum amplitude, the membrane does not touch the upper plate of the device (mathematically, it is represented by a singularity). Then, considering $K(x, u(x)) = |u''(x)|/(1 + |u'(x)|^2)^{-3/2}$ as the usual one-dimensional Cartesian curvature [14], problem (10) is rewritten as:

$$\begin{cases} u''(x) = -\frac{(1+(u'(x))^2)^3}{\beta_1 \mu^2(x,u(x),\lambda)} \quad \text{in} \quad \Omega \\ u(-L_1) = u(L_1) = 0 \\ 0 < u(x) < 1 - d^* = \alpha. \end{cases} \tag{13}$$

It is worth noting that model (13) represents a particular occurrence of the following general problem

$$\begin{cases} u''(x) + f(x, u(x), u'(x)) = 0 \text{ in } \Omega = [L_1, -L_1] \\ u(-L_1) = u(L_1) = 0 \\ 0 < u < \alpha \quad u \in C^2(\Omega) \end{cases} \tag{14}$$

in the Dirichlet's form, where f and α are characterized as $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R})$ and $\alpha = 1 - d^*$. Then:

$$\begin{cases} u''(x) = -\frac{(1+(u'(x))^2)^3}{\theta \mu^2(x,u(x),\lambda)} = -\frac{1}{\theta \lambda^2}(1 + (u'(x))^2)^3(\alpha - u(x))^2 \text{ in } \Omega \\ u(-L_1) = u(L_1) = 0; \quad 0 < u < \alpha \end{cases} \tag{15}$$

where $u \in C^2(\Omega)$ because membrane tears are not allowed, and the slopes continuously vary: $\mu = \mu(x, u(x), \lambda) \in C^0(\Omega \times [0, 1], [\bar{\lambda}, M])$ and $\mu = \lambda(\alpha - u(x))^{-1}$. So, the problem under study assumes the following form [13]:

$$\begin{cases} u''(x) = -\frac{(1+(u'(x))^2)^3}{\theta \mu^2(x,u(x),\lambda)} = -\frac{1}{\theta \lambda^2}(1 + (u'(x))^2)^3(\alpha - u(x))^2 \text{ in } \Omega \\ u(-L_1) = u(L_1) = 0 \\ 0 < u < \alpha \\ \mu = \mu(x, u(x), \lambda) \in C^0(\Omega \times [0, 1], [\bar{\lambda}, M]) \\ \mu = \frac{\lambda}{(\alpha - u(x))}. \end{cases} \tag{16}$$

Obviously, increasing $\theta\lambda^2$, $u''(x)$ decreases that, from the geometric point of view, represents the concavity of the membrane profile (presence of the minus sign). In other words, the higher the value of $\theta\lambda^2$, the more the membrane flattens out.

3. A well-known result of existence

The condition of existence for the solution of (15) obtained in [13] used a fixed-point result, according to which a completely continuous operator, which is defined by a convex, closed and limited subsets of a Banach space in itself, has at least one fixed point (Schauder–Tychonoff). Thus, indicating by $H = sup\{|u'(x)|\}$, two functional spaces P and P_1 were defined in $\Omega = [-L_1, L_1]$ (closed and limited sets on which the problem under study is defined):

$$P = \{C_0^2(\Omega) : \ 0 < u(x) < \alpha, \ |u'(x)| < H < +\infty\} \tag{17}$$

$$P_1 = \{C_0^1(\Omega) : \ 0 < u(x) < \alpha, \ |u'(x)| < H < +\infty\} \tag{18}$$

By differentiation and considering a suitable Green's function $G(x, s)$ [14], the problem is rewritten in its equivalent integral formulation:

$$u(x) = \int_{-L_1}^{L_1} G(x, s) f(s, u(s), u'(s)) ds, \quad 0 < u < \alpha \tag{19}$$

so that (15) assumes the form:

$$u(x) = \int_{-L_1}^{L_1} G(x, s) \frac{(1 + (u'(s))^2)^3}{\theta \mu^2(s, u(s), \lambda)} ds. \tag{20}$$

With these premises, the existence of the solution for $T(u) = w$ with $u \in P$ is proven by the fixed point approach, with T operator from P to P. For the details and proof, see [13]. The condition guarantees the existence of at least one solution to the problem under study depending on H and the parameters related to the physical properties of the membrane material of the device (including the electrostatic potential of inertia of the membrane: $\bar{\lambda}^2$). Formally, this condition was [13]:

$$1 + H^6 < \frac{H\theta\bar{\lambda}^2}{4\alpha L_1}. \tag{21}$$

These important results guarantee the applicability of the fixed-point approach. (21) is valid for each value of H, even for its upper value, $sup\{H\}$. However, a clarification is, at this point, a duty. In [13] the conditions of existence and uniqueness of the solution to the problem, studied numerically, showed a value of $sup\{H\} = sup\{|u'(x)|\} = 99$, corresponding to 88.92 degree dimensionless. This value is considerably high due to the fact that, in order to obtain (21), a large number of increases have been exploited, producing the value of 99 which, although excessive, is analytically correct.

4. On the uniqueness of the solution: A new result depending on the electro-mechanical characteristics of the membrane

[13] provided a proof of uniqueness of the solution[2] of the problem under study, which did not depend on the electro-mechanical properties of the membrane; thus, it seemed natural to propose here a new demonstration to derive a condition of uniqueness that depends on the type of membrane material. The theorem in this section proves that problem (15) admits a unique solution under a particular condition depending on the material of the membrane. Thus, let us consider, by contradiction, two different solutions of P: u_1, u_2. Before obtaining the condition to assure the uniqueness of the solution, we begin with two useful remarks.

Remark 4.1.

$$\left|(1 + (u_2')^2)^3 - (1 + (u_1')^2)^3\right| \leq 24H^5|u_2' - u_1'|. \tag{22}$$

[2] The proof presented in [13] used both Poincaré inequality and Gronwall Lemma.

Proof. Since $H > 1$, quantity $\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right|$ admits the following chain of inequalities:

$$\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right| = \left|[(1+(u_2')^2) - (1+(u_1')^2)][(1+(u_1')^2)^2 \right. \tag{23}$$
$$\left. + (1+(u_2')^2)(1+(u_1')^2) + (1+(u_2')^2)^2]\right|$$
$$= \left|[(u_2')^2 - (u_1')^2][(1+(u_1')^2)^2 + (1+(u_2')^2)(1+(u_1')^2) + (1+(u_2')^2)^2]\right|$$
$$= \left|[(u_2' - u_1')(u_2' + u_1')][(1+(u_1')^2)^2 + (1+(u_2')^2)(1+(u_1')^2) + (1+(u_2')^2)^2]\right|$$
$$\le |u_2' - u_1'| 2H[(1+H^2)^2 + (1+H^2)(1+H^2) + (1+H^2)^2]$$
$$= |u_2' - u_1'| 2H|(1+H^2)^2 + (1+H^2)^2 + (1+H^2)^2|$$
$$= |u_2' - u_1'| 2H|3(1+H^2)^2| = |u_2' - u_1'|(6H(1+H^2)^2)$$
$$= |u_2' - u_1'|(6H + 6H^5 + 12H^3) \le 24H^5|u_2' - u_1'|.$$

Remark 4.2. Considering that $\alpha < 1$ because $0 < u < 1-d^*, \forall\, u_1, u_2 \in P$, we can write:

$$\left|(1+(u_2')^2)^3(\alpha - u_2)^2 - (1+(u_1')^2)^3(\alpha - u_1)^2\right| \tag{24}$$
$$\le 216H^5|u_2' - u_1'| + 24(1+H^6)|u_2 - u_1|.$$

Proof. Considering the quantity $\left|(1+(u_2')^2)^3(1-d^*-u_2)^2 - (1+(u_1')^2)^3(1-d^*-u_1)^2\right|$, we can write:

$$\left|(1+(u_2')^2)^3(1-d^*-u_2)^2 - (1+(u_1')^2)^3(1-d^*-u_1)^2\right| \tag{25}$$
$$= \left|(1+(u_2')^2)^3(1+d^*+u_2^2 - 2d^* - 2u_2 + 2u_2 d^*)\right.$$
$$\left. - (1+(u_1')^2)^3(1+d^*+u_1^2 - 2d^* - 2u_1 + 2u_1 d^*)\right|$$
$$= \left|(1+(u_2')^2)^3 + d^*(1+(u_2')^2)^3 + u_2^2(1+(u_2')^2)^3 - 2d^*(1+(u_2')^2)^3\right.$$
$$- 2u_2(1+(u_2')^2)^3 + 2u_2 d^*(1+(u_2')^2)^3 - (1+(u_1')^2)^3 - d^*(1+(u_1')^2)^3$$
$$\left. - u_1^2(1+(u_1')^2)^3 + 2d^*(1+(u_1')^2)^3 + 2u_1(1+(u_1')^2)^3 - 2u_1 d^*(1+(u_1')^2)^3\right|$$
$$\le \left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right| + |u_2^2|\left|(1+(u_2')^2)^3 - u_1^2(1+(u_1')^2)^3\right|$$
$$+ d^*\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right| + 2d^*\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right|$$
$$+ 2\left|u_2(1+(u_2')^2)^3 - u_1(1+(u_1')^2)^3\right| + 2d^*\left|u_2(1+(u_2')^2)^3 - u_1(1+(u_1')^2)^3\right|$$
$$= \left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right| + \left|u_2^2(1+(u_2')^2)^3 - u_2^2(1+(u_1')^2)^3\right.$$
$$\left. + u_2^2(1+(u_1')^2)^3 - u_1^2(1+(u_1')^2)^3\right| +$$
$$+ d^*\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right| + 2d^*\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right|$$
$$+ 2\left|u_2(1+(u_2')^2)^3 - u_2(1+(u_1')^2)^3 + u_2(1+(u_1')^2)^3 - u_1(1+(u_1')^2)^3\right|$$
$$+ 2d^*\left|u_2(1+(u_2')^2)^3 - u_2(1+(u_1')^2)^3 + u_2(1+(u_1')^2)^3 - u_1(1+(u_1')^2)^3\right|$$
$$\le \left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right| + |u_2^2|\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right|$$
$$+ |u_2^2 - u_1^2|(1+(v_1')^2)^3 + (d^* + 2d^*)\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right|$$
$$+ (2+2d^*)\left|u_2(1+(u_2')^2)^3 - u_2(1+(u_1')^2)^3\right.$$
$$\left. + u_2(1+(u_1')^2)^3 - u_1(1+(u_1')^2)^3\right|$$
$$\le (1 + |u_2^2| + (d^* + 2d^*) + (2+2d^*)|u_2|)\left|(1+(u_2')^2)^3 - (1+(u_1')^2)^3\right|$$
$$+ [|u_2 - u_1|(2+2d^*) + |u_2^2 - u_1^2|](1+(u_1')^2)^3$$
$$\le 216H^5|u_2' - u_1'| + |u_2 - u_1|[(2+2d^*) + |u_2 + u_1|](1+(u_1')^2)^3$$
$$\le 216H^5|u_2' - u_1'| + 24(1+H^6)|u_2 - u_1|$$

Based on these premises, we should enunciate the following theorem, where we present the new condition[3] guarantees the uniqueness of the solution for (15). Formally, the following theorem for uniqueness holds.

Theorem 4.3. *If*

$$1 + H^6 < \frac{\theta\lambda^2}{24L_1(L_1+1)} \tag{26}$$

Then, the problem under study admits the uniqueness of the solution.

Proof. By contradiction, we assume that $u_1, u_2 \in P$ are two different solutions for the problem under study, so $u_1 = T(u_1)$ e $u_2 = T(u_2)$. Thus, we can write:

$$u_1(x) = \int_{-L_1}^{L_1} G(x,s)\frac{(1+(u_1'(s))^2)^3}{\theta\mu^2(s,u_1(s),\lambda)}ds \tag{27}$$
$$= \int_{-L_1}^{L_1} \frac{1}{\theta\lambda^2} G(x,s)(1+(u_1'(s)^2)^3)(\alpha - u_1(s))^2 ds$$

$$u_2(x) = \int_{-L_1}^{L_1} G(x,s)\frac{(1+(u_2'(s))^2)^3}{\theta\mu^2(s,u_2(s),\lambda)}ds \tag{28}$$
$$= \int_{-L_1}^{L_1} \frac{1}{\theta\lambda^2} G(x,s)(1+(u_2'(s)^2)^3)(\alpha - u_2(s))^2 ds$$

$$u_1'(x) = \int_{-L_1}^{L_1} G_x(x,s)\frac{(1+(u_1'(s))^2)^3}{\theta\mu^2(s,u_1(s),\lambda)}ds \tag{29}$$
$$= \int_{-L_1}^{L_1} \frac{1}{\theta\lambda^2} G_x(x,s)((1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 ds$$

$$u_2'(x) = \int_{-L_1}^{L_1} G_x(x,s)\frac{(1+(u_2'(s))^2)^3}{\theta\mu^2(s,u_2(s),\lambda)}ds \tag{30}$$
$$= \int_{-L_1}^{L_1} \frac{1}{\theta\lambda^2} G_x(x,s)((1+(u_2'(s))^2)^3)(\alpha - u_2(s))^2 ds$$

Therefore, we have:

$$\|u_1 - u_2\|_{C^1([-L_1,L_1])} = sup_{x\in[-L_1,L_1]}|u_1 - u_2| + sup_{x\in[-L_1,L_1]}|u_1' - u_2'| \tag{31}$$

Considering (23), we can write:

$$\|T(u_1) - T(u_2)\| = \frac{1}{\theta\lambda^2} sup_{x\in[-L_1,L_1]} \tag{32}$$
$$\times \left| \int_{-L_1}^{L_1} G(x,s)((1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 ds \right.$$
$$\left. - \int_{-L_1}^{L_1} G(x,s)((1+(u_2'(s))^2)^3)(\alpha - u_2(s))^2 ds \right|$$
$$+ \frac{1}{\theta\lambda^2} sup_{x\in[-L_1,L_1]}\left| \int_{-L_1}^{L_1} G_x(x,s)((1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 ds \right.$$
$$\left. - \int_{-L_1}^{L_1} G_x(x,s)((1+(u_2'(s))^2)^3)(\alpha - u_2(s))^2 ds \right|$$
$$= \frac{1}{\theta\lambda^2} sup_{x\in[-L_1,L_1]}\left| \int_{-L_1}^{L_1} G(x,s)[((1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 \right.$$
$$\left. + ((1+(u_2'(s))^2)^3)(\alpha - u_2(s))^2] ds \right|$$
$$+ \frac{1}{\theta\lambda^2} sup_{x\in[-L_1,L_1]}\left| \int_{-L_1}^{L_1} G_x(x,s)[-(1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 \right.$$
$$\left. + ((1+(u_2'(s))^2)^3)(\alpha - u_2(s))] ds \right|$$
$$\le \frac{1}{\theta\lambda^2}\frac{L_1}{2} sup_{x\in[-L_1,L_1]}\left| \int_{-L_1}^{L_1} [(-(1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 \right.$$
$$\left. + ((1+(u_2'(s))^2)^3)(\alpha - u_2(s))^2] ds \right|$$
$$+ \frac{1}{2\theta\lambda^2} sup_{x\in[-L_1,L_1]}\left| \int_{-L_1}^{L_1} [(-(1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 \right.$$
$$\left. + ((1+(u_2'(s))^2)^3)(\alpha - u_2(s))] ds \right|$$
$$= \frac{1}{\theta\lambda^2}\left(\frac{L_1}{2} + \frac{1}{2}\right) sup_{x\in[-L_1,L_1]}\left| \int_{-L_1}^{L_1} [(-(1+(u_1'(s))^2)^3)(\alpha - u_1(s))^2 \right.$$
$$\left. + ((1+(u_2'(s))^2)^3)(\alpha - u_2(s))] ds \right|$$

Considering (25), we can write:

$$\|T(u_1) - T(u_2)\|_{C^1([-L_1,L_1])} \tag{33}$$
$$\le \frac{1}{\theta\lambda^2}\left(\frac{L_1}{2} + \frac{1}{2}\right) sup_{x\in[-L_1,L_1]}$$

[3] With respect to [13].

$$\times \left| \int_{-L_1}^{L_1} (216H^5|u_2' - u_1'| + 24(1+H^6)|u_2 - u_1|) ds \right|$$
$$= \frac{1}{\theta\lambda^2}\left(\frac{L_1}{2} + \frac{1}{2}\right)(216H^5 2L_1) sup_{s\in[-L_1,L_1]}|u_2'(s) - u_1'(s)|$$
$$+ \frac{1}{\theta\lambda^2}\left(\frac{L_1}{2} + \frac{1}{2}\right)(24(1+H^6)2L_1) sup_{s\in[-L_1,L_1]}|u_2(s) - u_1(s)|.$$

However, being $u_1 = T(u_1)$ and $u_2 = T(u_2)$, by (31), with (33), we obtain a contradiction if:

$$\begin{cases} \frac{2L_1}{\theta\lambda^2}\left(\frac{L_1}{2} + \frac{1}{2}\right)216H^5 < 1 \\ \frac{2L_1}{\theta\lambda^2}\left(\frac{L_1}{2} + \frac{1}{2}\right)24(1+H^6) < 1 \end{cases} \tag{34}$$

that is:

$$\begin{cases} \frac{1}{\theta\lambda^2} L_1(L_1+1)216H^5 < 1 \\ \frac{1}{\theta\lambda^2} L_1(L_1+1)24(1+H^6) < 1 \end{cases} \tag{35}$$

and again:

$$\begin{cases} 216H^5 < \frac{\theta\lambda^2}{L_1(L_1+1)} \\ 24(1+H^6) < \frac{\theta\lambda^2}{L_1(L_1+1)}. \end{cases} \tag{36}$$

Considering the first inequality of (36), we can also write:

$$H^6 = H^5 H < \frac{\theta\lambda^2}{216L_1(L_1+1)} H \tag{37}$$

from which:

$$1 + H^6 < 1 + \frac{\theta\lambda^2 H}{216L_1(L_1+1)} \tag{38}$$

so (36) assumes the following form:

$$\begin{cases} 1 + H^6 < 1 + \frac{\theta\lambda^2 H}{216L_1(L_1+1)} \\ 1 + H^6 < \frac{\theta\lambda^2}{24L_1(L_1+1)}. \end{cases} \tag{39}$$

Furthermore, we observe that:

$$\frac{\theta\lambda^2}{24L_1(L_1+1)} < 1 + \frac{\theta\lambda^2 H}{216L_1(L_1+1)} \tag{40}$$

in fact, starting from (40), we can write:

$$9\theta\lambda^2 < 216L_1(L_1+1) + \theta\lambda^2 H \tag{41}$$

from which

$$9 < \frac{216}{\theta\lambda^2} L_1(L_1+1) + H \tag{42}$$

and again:

$$H > 9 - \frac{216}{\theta\lambda^2} L_1(L_1+1) \tag{43}$$

so that

$$H > 9\left(1 - \frac{24}{\theta\lambda^2} L_1(L_1+1)\right). \tag{44}$$

That is definitely true for the competition. In fact, if we suppose by contradiction that

$$\frac{\theta\lambda^2}{24L_1(L_1+1)} > 1 + \frac{\theta\lambda^2 H}{216L_1(L_1+1)} \tag{45}$$

we can write

$$9 > \frac{216}{\theta\lambda^2} L_1(L_1+1) + H \tag{46}$$

from which:

$$H < 9 - \frac{216}{\theta\lambda^2} L_1(L_1+1) < 0. \tag{47}$$

In other words, $H = sup|u'|$ assumes a negative value that represents an impossible condition. Therefore, (34) is equivalent to the following inequality:

$$1 + H^6 < \frac{\theta\lambda^2}{24L_1(L_1+1)} \tag{48}$$

which is a representative of the constraint ensuring the uniqueness of the solution for the problem under study. Even the uniqueness depends on the physical parameters of the device's membrane, but the inertia $\bar{\lambda}^2$ of the membrane does not appear, which confirms the experimental fact that when voltage V is applied to the device, the membrane moves if V overcomes the inertia $\bar{\lambda}^2$. Therefore, the condition of existence of the solution (21) depends on $\bar{\lambda}^2$. However, when the membrane has moved by overcoming its inertia, the condition that guarantees the uniqueness of the solution (26) is independent of $\bar{\lambda}^2$.

5. A condition ensuring both existence and uniqueness

Considering that (21) ensures the existence of the solution for the problem under study and (26) ensures its uniqueness, it is imperative to solve the following system:

$$\begin{cases} 1 + H^6 < \frac{H\theta\bar{\lambda}^2}{4\alpha L_1} \quad (existence) \\ 1 + H^6 < \frac{\theta\lambda^2}{24L_1(L_1+1)} \quad (uniqueness). \end{cases} \tag{49}$$

Immediately, we observe that the following condition is true:

$$\frac{\theta\lambda^2}{24L_1(L_1+1)} < \frac{H\theta\bar{\lambda}^2}{4\alpha L_1} \tag{50}$$

In fact, supposing for absurdity that:

$$\frac{1}{24(L_1+1)} > \frac{1}{4\alpha} H\bar{\lambda}^2 \tag{51}$$

we obtain

$$H < \frac{\alpha}{6(L_1+1)\bar{\lambda}^2} \tag{52}$$

and considering that $L_1 = 0.5$, we can write:

$$H < \frac{\alpha}{9\bar{\lambda}^2}. \tag{53}$$

Finally, considering the scaling factors and condition (53), it is permissible to write:

$$H = \frac{z}{x} = \frac{z' 2L_1}{hx'} < \frac{\alpha}{9\bar{\lambda}^2} \tag{54}$$

Hence, if we denote the dimensionless value of H by H', the following holds:

$$\frac{H' 2L_1}{h} < \frac{\alpha}{9\bar{\lambda}^2} \tag{55}$$

so that:

$$H' < \frac{h\alpha}{18L_1\bar{\lambda}^2}. \tag{56}$$

Thus, H' should be lower than a notably small amount that is incompatible with the definition of H'. Therefore, system (49) is equivalent to the unique inequality:

$$1 + H^6 < \frac{\theta\lambda^2}{24L_1(L_1+1)} \tag{57}$$

which represents the new condition to be imposed to H to ensure both existence and uniqueness of the solution for the problem under study. In other words, the uniqueness of the solution is a guarantee of its existence unlike what obtained in [13] where it was necessary to evaluate the existence of the solution while the uniqueness was still guaranteed. This was due to the fact that uniqueness had been demonstrated independently of the characteristics of the material constituting the membrane with consequent reduction of the risk of obtaining ghost solutions (that is numerical solutions not verifying the analytical condition of existence). Finally, it is worth underlining the fact that in (57) the α parameter does not appear for which the existence and uniqueness of the solution to the problem under study is independent of the safety distance.

6. Some numerical tests

6.1. The exploited numerical approach

In [3] the model under study (16) was already dealt numerically by a shooting procedure by rewriting the model (16) as a first-order system as follows

$$\begin{cases} u_1'(x) = u_2(x) \\ u_2'(x) = -\frac{1}{\theta\lambda^2}(1+u_2(x)^2)^3(\alpha_1 - u_1(x))^2 \end{cases} \tag{58}$$

in which

$$\begin{cases} u_1(x) = u(x) \\ u_2(x) = \frac{du_1(x)}{dx} = \frac{du(x)}{dx}. \end{cases} \tag{59}$$

The model (58) (BVP) was then transformed into an IVP equivalent by replacing the boundary condition of the solution $u_1(L_1)$ at $x = L_1$ with $u_2(-L_1) = \eta$, $\eta \in \mathbb{R}$, as initial condition defining implicitly a nonlinear equation $F(\eta) = u_1(L_1;\eta) = 0$. The latter, starting from suitable values of both η_0 and η_1, the system has been solved numerically,[4] by means of secant method,[5] method generating a sequence of values η_k, $k = 2, 3, \ldots$, until $\eta_k \to \eta$, $k \to +\infty$ satisfying a suitable termination criteria (for details, see [3]).

6.2. Some results of convergence

With these premises, in [3], $\theta\lambda^2$ values which guaranteed convergence were all and only those for which $\theta\lambda^2 \in [0.63, +\infty)$ in the sense that values of $\theta\lambda^2 \in [0, 0.63)$ did not guarantee convergence. So, indicating with $(\theta\lambda^2)_{conv} \in [0.63, +\infty)$, the following chain of inequality made sense:

$$inf\{(\theta\lambda^2)_{conv}\} = min\{(\theta\lambda^2)_{conv}\} = 0.63 \tag{60}$$

which generated, as d^*, the merit curves ($\theta\lambda^2$ vs H) in which the areas of convergence and non-convergence were evident. Furthermore, by superimposing the curves obtained numerically ($\theta\lambda^2$ vs H_j, $j \in \mathbb{N}$ jth profiled obtained in condition of convergence) with the analytical curve ($\theta\lambda^2$ vs H) the values of $\theta\lambda^2$ that generated ghost solutions were highlighted (that is all the solutions obtained numerically that did not satisfy the analytical condition of existence and uniqueness (21)). Figs. 1 and 2 show these results achieved in [3] for a particular value of d^*.

6.3. Search for eventual ghost solutions

Since the model under study (16) is the same as that studied in [13], the range of convergence is the same ($\theta\lambda^2 \in [0.63, +\infty)$). In addition, the equalities (60) hold and the profiles of the membrane achieved numerically in [13] as solutions of the model (16) are still valid. However, changing the condition that establishes the existence and uniqueness of the solution (that is passing from (21) to (48)) the figures of merit above defined change and, consequently, the range $\theta\lambda^2$, in which the ghost solutions fall back, changes. And again, from the analytical condition of existence and uniqueness of the solution obtained here in this work, (57), we can write:

$$H < \sqrt[6]{\frac{\theta\lambda^2}{24L_1(L_1+1)}} - 1 \tag{61}$$

that makes sense if only if the rooting is not negative:

$$\frac{\theta\lambda^2}{24L_1(L_1+1)} - 1 \geq 0 \tag{62}$$

[4] By means of ODE23 solver, MatLab ODE suite with the accuracy and adaptivity parameters defined by default. The routines have been implemented on Intel Core 5 CPU 1.47 GHz using MatLab R2013a.

[5] Because Newton method required more restrictive convergence conditions.

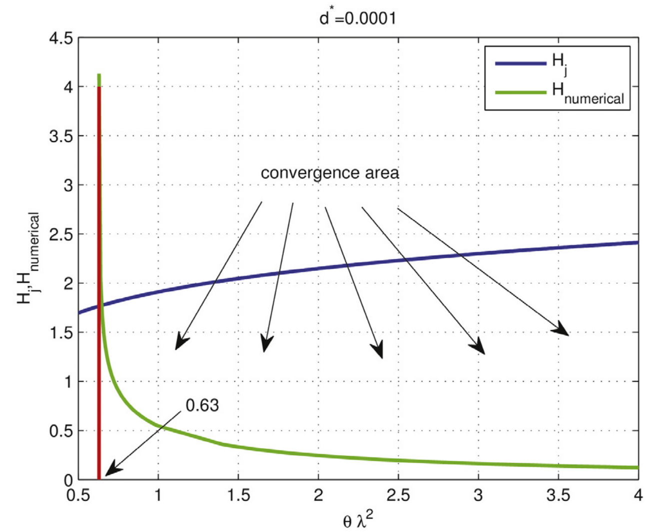

Fig. 1. Convergence area achieved in [3] with $d^* = 0.0001$.

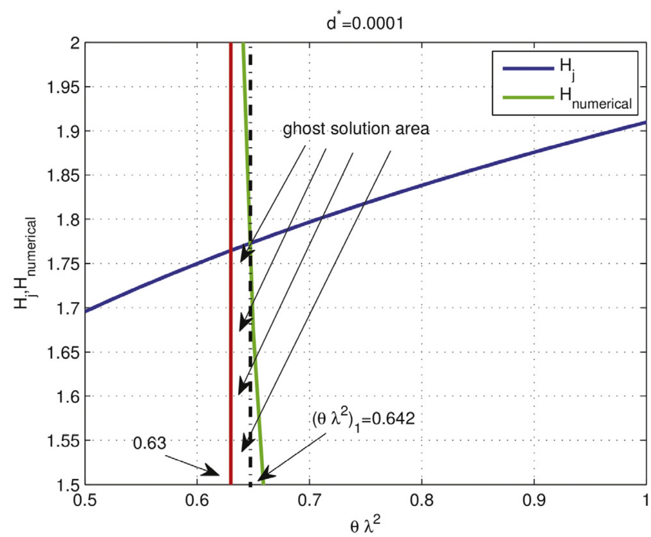

Fig. 2. Ghost solutions area highlighted in [3] with $d^* = 0.0001$.

from which, setting $L_1 = 0.5$, we obtain:

$$\theta\lambda^2 \geq 24L_1(L_1+1) = 18. \tag{63}$$

This means that for $\theta\lambda^2 \in [0, 0.63)$ the convergence of the shooting procedure is not guaranteed, whereas for $\theta\lambda^2 \in [18, +\infty)$, in addition to being in conditions of convergence of the numerical procedure, we are sure not to have ghost solutions and, increasing $\theta\lambda^2$, increases the value of H eliminating the limitation $H = 99$ that in [13] took place.[6] Finally, when $\theta\lambda^2 \in [0.63, 18)$, despite being in conditions of convergence, we are in the presence of ghost solutions since, being $H < 0$, we are in the presence of an unachievable physical condition because, although subjecting the device to a given electrical voltage, the membrane deforms in the opposite direction. (See Fig. 3.)

6.4. Some considerations on the fields of application of the device

From (61) it is possible to obtained the range of the values of V that the device here presented admits in condition of convergence and in the

[6] Even if, as above mentioned, $H = 99$ is a sufficiently high value.

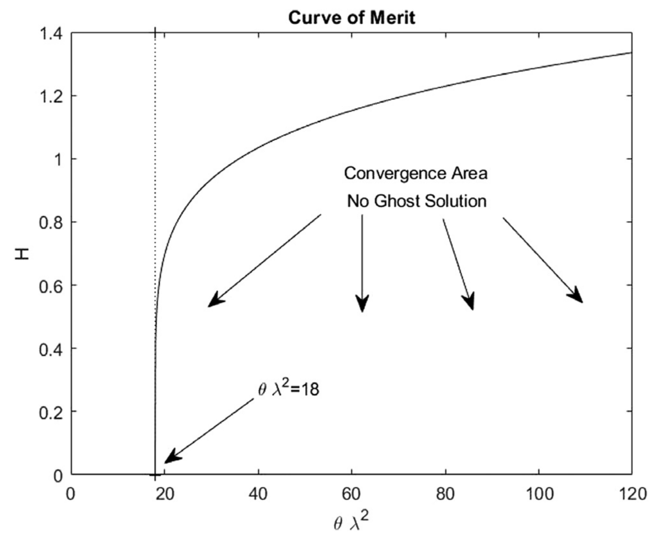

Fig. 3. New curve of merit H vs $\theta\lambda^2$ (the dependence on d^* is not present). If $\theta\lambda^2 \geq 18$ the convergence of the numerical method is guarantee in absence of ghost solutions.

absence of ghost solutions. In fact, taking into account that [13]:

$$\lambda^2 = \theta V^2 \tag{64}$$

from (57), and indicating with $(\theta\lambda^2)_{inf}$ the inf of $\theta\lambda^2$, we can write:

$$1 + H^6 < \frac{(\theta\lambda^2)_{inf}}{24L_1(L_1+1)} < \frac{\theta\lambda^2}{24L_1(L_1+1)} = \frac{\theta^2 V^2}{18} \tag{65}$$

from which:

$$V > \sqrt{\frac{18(1+H^6)}{\theta^2}} \tag{66}$$

that represents, once chosen the material constituting the membrane (parameter θ), the permissible values of V to which the device can undergo convergence conditions (and in the absence of ghost solutions). Conversely, from (65):

$$H < \sqrt[6]{\frac{\theta^2 V^2}{18} - 1} \tag{67}$$

that, once chosen the material constituting the membrane and varying the applied voltage V, fixes he range of admissible values of H. In addition, since the condition [3,13]

$$\theta E^2 = \frac{\lambda^2}{(1-u(x))^2} = \frac{\epsilon_0 V^2 (2L)^2}{2h^3(1-u(x))^2} = \frac{\beta V^2}{(1-u(x))^2} \tag{68}$$

is still valid and, multiplying (68) by λ^2 and considering that $(1-u(x))^2 < 1$, $E^2 < sup\{E^2\}$, $\beta_1 = \frac{\epsilon_0}{2\sigma}$ and $\lambda^2 = \beta_1 V^2$, the two following expressions to be valid [3]:

$$\theta\lambda^2 E^2 = \frac{\beta_1 V^2 \lambda^2}{(1-u(x))^2} = \frac{\beta_1^2 V^4}{(1-u(x))^2} \tag{69}$$

$$\theta\lambda^2 = \frac{\epsilon_0^2 V^2}{4\sigma^2(1-u(x))^2 E^2} > \frac{\epsilon_0^2 V^4}{4\sigma^2 sup\{E^2\}} \tag{70}$$

so that, in convergence conditions and in absence of ghost solutions, the electromechanical characteristics of the membrane (σ) the operation parameters (V, $sup\{E^2\}$) verify:

$$\frac{\epsilon_0^2 V^4}{4\sigma^2 sup\{E^2\}} \geq 18. \tag{71}$$

Then, taking into account both (66) and (71), we can write:

$$\begin{cases} \sigma \leq \frac{\epsilon V^2}{2\sqrt{18 sup\{E^2\}}} \\ V > \sqrt{\frac{18(1+H^6)}{\theta^2}} \end{cases} \tag{72}$$

which represent the two conditions to which σ, V and E must meet to ensure both convergence of the numerical method and absence of ghost solutions. Again, from (67), setting $\theta = 10^{12}$ and $V = 1$[7] we can write:

$$H < \sqrt[6]{\frac{10^{12}}{18} - 1} \simeq 62 \tag{73}$$

corresponding to 88.61 degree (in dimensionless conditions) which still represents a good limitation for $sup|H|$. Then, the limitations (72) can be furthermore written as:

$$\begin{cases} \sigma \leq \frac{\epsilon V^2}{2\sqrt{18 sup\{E^2\}}} \\ V > \sqrt{\frac{18(1+sup|H|^6)}{\theta^2}} = \sqrt{\frac{18(1+62^6)}{10^{12}}} = 2.63 \cdot 10^{-4}\ \text{V}. \end{cases} \tag{74}$$

However, even with these limitations,[8] all numerical results in [3] remain valid, which confirms that the methods based on shooting techniques can efficiently obtain stable solutions of the model under study.

7. Conclusion and perspectives

In this work, the authors propose a method to improve the conditions for the existence and uniqueness of the solution of a well-known $1D$ differential model of a membrane MEMS device where the contribution from the amplitude of the electrostatic field, $|\mathbf{E}|$, in the device (following the application of the external voltage, V) was thought to be locally proportional to the membrane curvature, K. The model was examined through a Schauder–Tychonoff's fixed-point approach provided a condition for the existence of a solution depending on the electromechanical characteristics of the membrane material, as required by innumerable industrial applications. However, for the uniqueness of the solution, such adherence with the application industrialists has not been manifested. Thus, starting from the aforementioned works, the authors have highlighted an alternative method to demonstrate uniqueness of the solution and obtain a new condition that is explicitly linked to the geometrical characteristics of the device and electromechanical properties of the membrane. Combined with the condition of existence, this condition shows, exploiting a numerical approach based on shooting techniques, a good adherence with the experimental results, which imperatively require that this condition is linked to the type of MEMS device. Moreover, the present study has highlighted a new range of values of electromechanical characteristics of the membrane and operational parameters more adherent to the experimentation that guarantee the convergence of the numerical approach and, at the same time, avoid the presence of ghost solutions. However, to improve the quality of the results, in the future, it appears important to make more performant the mathematical model considering more sophisticated geometrical curvatures formulations, even if it requires higher regularity conditions.

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