NATURAL FREQUENCIES OF STRUCTURES WITH INTERVAL PARAMETERS

A. Sofi[1) *], G. Muscolino[2)], I. Elishakoff[3)]

[1]*Department of Civil, Energy, Environmental and Materials Engineering, University "Mediterranea" of Reggio Calabria, Via Graziella, Località Feo di Vito, 89124 Reggio Calabria, Italy; e-mail: alba.sofi@unirc.it*

[2]*Department of Civil, Building and Environmental Engineering with Information Technology and Applied Mathematics, University of Messina, Villaggio S. Agata, 98166 Messina, Italy; e-mail: gmuscolino@unime.it*

[3]*Department of Ocean and Mechanical Engineering, Florida Atlantic University, 77 Glades Road, 33431 Boca Raton, Florida, USA.E-mail: elishako@fau.edu*

Abstract

This paper deals with the evaluation of the lower and upper bounds of the natural frequencies of structures with uncertain-but-bounded parameters. The solution of the generalized interval eigenvalue problem is pursued by taking into account the actual variability and dependencies of uncertain structural parameters affecting the mass and stiffness matrices. To this aim, interval uncertainties are handled by applying the *improved interval analysis via extra unitary interval* (*EUI*), recently introduced by the first two authors. By associating an *EUI* to each uncertain-but-bounded parameter, the cases of mass and stiffness matrices affected by fully disjoint, completely or partially coincident uncertainties are considered. Then, based on sensitivity analysis, it is shown that the bounds of the interval eigenvalues can be evaluated as solution of two appropriate deterministic eigenvalue problems without requiring any combinatorial procedure. If the eigenvalues are monotonic functions of the uncertain parameters, then the exact bounds are obtained. The accuracy of the proposed method is demonstrated by numerical results concerning truss and beam structures with material and/or geometrical uncertainties.

Keywords: Interval uncertainties; Generalized interval eigenvalue problem; Interval natural frequencies; Improved interval analysis; Extra unitary interval; Sensitivity analysis; Eigenvalue bounds.

*Corresponding author

1. INTRODUCTION

The evaluation of the natural frequencies and the corresponding mode shapes plays a crucial role in vibration analysis since it provides a great deal of information concerning the dynamic characteristics of a system. Within a deterministic setting, this task is accomplished by solving the generalized eigenvalue problem which involves the mass and stiffness matrices of the structure. Changes of inertial and stiffness properties due to uncertainties inherent in any design process may affect to a large extent the vibration characteristics of a structural system. It is, therefore, of primary interest for design purposes to estimate the effects of geometrical and/or material uncertainties on the natural frequencies. Such uncertainties are commonly described within a probabilistic framework by using the random variable or random field concept. However, in the last decades, the so-called non-probabilistic approaches, such as convex model, fuzzy sets or interval model [1], have increasingly spread as alternative tools for handling uncertainties arising in engineering problems. The interval model, stemming from the *interval analysis* [2,3], is widely used when only the range of variability of non-deterministic properties is known but available data are insufficient to make reliable assumptions on the joint probability density function.

If the uncertain parameters are modeled as interval variables, the mass and stiffness matrices of the structure turn out to be interval matrices and the eigenvalue analysis leads to the so-called generalized or standard interval eigenvalue problems. The solution of these problems is a very difficult task since it consists in the evaluation of all possible eigenvalues and eigenvectors as the interval stiffness and mass matrices vary between their bounds. In practice, the objective is the determination of the narrowest intervals enclosing all possible eigenproperties, say the evaluation of the bounds of the eigenvalue and associated eigenvector for each eigensolution.

The solution of the interval eigenvalue problem has attracted much research attention in the last decades. Rohn [4] studied the generalized interval eigenvalue problem and derived formulas for the interval eigenvalues of a symmetric interval matrix with an error matrix of rank one. Based on the invariance properties of the characteristic vector entries, Deif [5] developed a method for the solution of the standard interval eigenvalue problem. The application of this method is limited by the lack of an efficient criterion for judging the invariance of signs of the eigenvectors components under interval matrix operations before computing interval eigenvalues. Under the assumption that the deviation amplitudes of the mass and stiffness matrices are positive semi-definite, Qiu *et al.* [6] proposed a procedure for the solution of the generalized interval eigenvalue problem which leads to two deterministic eigenvalue problems involving the bounds of the mass and stiffness matrices. The effectiveness of this method has been assessed by comparison with Deif's solution in the simplest case of fully disjoint mass and stiffness uncertainties. Following a similar reasoning, Elishakoff [7] proposed a procedure for finding the range of eigenvalues due to uncertain elastic moduli and mass density by using the upper and lower stiffness and mass matrices. A perturbation method for the solution of the generalized interval eigenproblem has been developed by Qiu *et al.* [8] by viewing the deviation amplitudes of the mass and stiffness matrices as perturbations around the nominal values of the interval matrix pair. The procedure is applicable for small deviation amplitudes and has been validated only in the case of fully disjoint mass and stiffness uncertainties. Qiu *et al.* [9] introduced the Eigenvalue Inclusion Principle (EIP) which leads to the solution of two deterministic eigenvalue problems as well. If the mass and stiffness are affected by different uncertainties, the exact bounds are obtained. In general, this approach is accurate and efficient but it does not provide a physically consistent treatment of uncertainties affecting simultaneously the stiffness and mass matrices. Furthermore, the EIP is applicable only when the matrix pairs can be expressed by the non-negative decomposition. Based on a previously developed interval finite element method, Modares *et al.* [10] proved that, in the presence of any physically allowable uncertainty in the structural stiffness, the solutions of two deterministic eigenvalue problems are sufficient to obtain the exact bounds of the system's fundamental frequencies without resorting to any combinatorial solution procedure. Gao [11] proposed the interval factor method to investigate the effects of geometrical and material interval uncertainties on the natural frequencies and mode shapes of truss structures. Despite its simplicity, the method provides physically inconsistent results such as the independency of natural frequencies and mode shapes on the uncertainty of cross-sectional areas and Young's moduli, respectively. Furthermore, the dispersion of the interval eigenproperties around their midpoint values turns out to be unexpectedly independent of the mode order. Several perturbation-based (see e.g. [12-15]) or iterative procedures (see e.g. [16-19]) for the evaluation of the interval eigenvalue bounds have been also developed in the last decades. An evolution strategy for computing eigenvalue bounds of interval matrices has been presented by Yuan *et al.* [20]. In an attempt to take into account the dependencies of the uncertain parameters entering the mass and stiffness matrices, recently an approach based on a modified affine arithmetic has been proposed [21]. Besides the involved solution procedure, a common drawback of the aforementioned approaches is that their accuracy is assessed only for simple examples with fully disjoint mass and stiffness uncertainties.

The aim of this paper is to propose an efficient method for the solution of the generalized interval eigenvalue problem, able to overcome the limitations of available procedures discussed above. The key idea is to seek the bounds of the eigenvalues taking into account the actual influence of uncertainties on the mass and stiffness matrices and their dependencies. In other words, rather than tackling the problem from a merely mathematical point of view, the proposed procedure seeks a solution consistent with the physical behaviour of the structure. Interval uncertainties are handled following the *improved interval analysis via extra unitary interval* [22,23]. All possible situations occurring in real engineering problems, where uncertainties affecting the mass and stiffness matrices may be fully disjoint, completely or partially coincident, are examined. In each of these cases, a preliminary sensitivity analysis is performed in order to investigate the behaviour of the eigenvalues as functions of the uncertain parameters [1,24]. Based on the information provided by the eigenvalue sensitivities, the combinations of the extreme values of the uncertain parameters corresponding to the bounds of the eigenvalues are determined. Hence, the eigenvalue bounds can be evaluated as solution of two appropriate deterministic eigenvalue problems without any combinatorial procedure. This ensures substantial computational advantages over the *vertex method* [25] which yields the exact bounds of monotonic eigenvalues at the expense of the onerous solution of as many deterministic eigenvalue problems as are the combinations of the extreme values of the uncertain structural parameters.

The accuracy of the proposed procedure is demonstrated by analysing two truss structures and a FE modeled cantilever beam in the three cases of mass and stiffness matrices affected by fully disjoint, completely coincident and partially coincident uncertainties. It is demonstrated that the proposed estimates of the eigenvalue bounds are exact as long as the eigenvalues are monotonic functions of the uncertain parameters.

2. PROBLEM FORMULATION

2.1 Interval uncertainty modeling via Extra Unitary Interval

The present study focuses on eigenvalue analysis of linear undamped structural systems with uncertain parameters, such as material and geometrical properties, affecting the mass and stiffness matrices. Within a non-probabilistic framework, uncertainties are represented as closed real interval numbers according to the so-called interval model. This model, mainly based on the *interval analysis* [2,3], turns out to be a very useful tool to carry out engineering analyses when only the range of variability of the uncertain parameters is available.

Denoting by $\mathbb{IR}$ the set of all closed real interval numbers, let $\boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}] \in \mathbb{IR}^r$ be a bounded set-interval vector of real numbers such that $\underline{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \overline{\boldsymbol{\alpha}}$. The apex *I* means interval variable, while the symbols $\underline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\alpha}}$ denote the lower bound (LB) and upper bound (UB) vectors. According to the *classical interval arithmetic*, the *i*-th real interval variable $\alpha_i^I = [\underline{\alpha}_i, \overline{\alpha}_i]$ is characterized by the midpoint value (or mean), $\alpha_{0,i}$, and the deviation amplitude (or radius), $\Delta\alpha_i$, given by:

$$\alpha_{0,i}=\frac{1}{2}\left(\underline{\alpha}_i+\overline{\alpha}_i\right);\quad \Delta\alpha_i=\frac{1}{2}\left(\overline{\alpha}_i-\underline{\alpha}_i\right). \tag{1a,b}$$

The real numbers $\alpha_i \in \alpha_i^I=\left[\underline{\alpha}_i,\overline{\alpha}_i\right]$, collected into the vector $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]$, are here assumed to represent the dimensionless fluctuations of the uncertain structural parameters.

Following the *improved interval analysis via extra unitary interval* [22,23], the i-th interval parameter α_i^I is here defined in the following *affine form*:

$$\alpha_i^I=\alpha_{0,i}+\Delta\alpha_i\hat{e}_i^I \tag{2}$$

where $\hat{e}_i^I \triangleq [-1,+1]$, $(i=1,2,\ldots,r)$, is the *extra unitary interval* (*EUI*) associated with α_i^I which satisfies the following properties:

$$\begin{aligned}&\hat{e}_i^I-\hat{e}_i^I=0;\quad \hat{e}_i^I\times\hat{e}_i^I=\left(\hat{e}_i^I\right)^2=[1,1];\quad \hat{e}_i^I/\hat{e}_i^I=[1,1];\\&\hat{e}_i^I\times\hat{e}_j^I=[-1,+1],\quad i\neq j;\quad x_i\hat{e}_i^I\pm y_i\hat{e}_i^I=\left(x_i\pm y_i\right)\hat{e}_i^I;\\&x_i\hat{e}_i^I\times y_i\hat{e}_i^I=x_iy_i\left(\hat{e}_i^I\right)^2=x_iy_i[1,1].\end{aligned} \tag{3a-f}$$

In these equations, $[1,1]=1$ is the so-called unitary *thin interval*. It is useful to remember that a thin interval occurs when $\underline{x}=\overline{x}$ and it is defined as $x^I\triangleq[\underline{x},\underline{x}]$, so that $x\in\mathbb{R}$.

In structural engineering problems, the dimensionless fluctuations of the uncertain-but-bounded parameters around their nominal values can be reasonably modeled as symmetric intervals, i.e. $\alpha_i^I=\left[\underline{\alpha}_i,\overline{\alpha}_i\right]$ with $\overline{\alpha}_i=-\underline{\alpha}_i$. Under this assumption, since $\alpha_{0,i}=0$ and $\Delta\alpha_i=-\underline{\alpha}_i=\overline{\alpha}_i$, Eq. (2) reduces to:

$$\alpha_i^I=\Delta\alpha_i\hat{e}_i^I. \tag{4}$$

Furthermore, to assure physically meaningful values of the uncertain structural properties, the deviation amplitudes $\Delta\alpha_i$ should satisfy the conditions $|\Delta\alpha_i|<1$, with the symbol $|\bullet|$ denoting absolute value. For instance, if the uncertain Young's modulus of the i-th structural element is expressed as $E_i^I = E_{0,i}\left(1+\Delta\alpha_i \hat{e}_i^I\right)$, with $E_{0,i}$ denoting the nominal value, the fluctuation α_i^I defined by Eq. (4) must satisfy the conditions $\left|\Delta\alpha_i\right| < 1$ to yield always positive values of the interval material property.

According to interval symbolism, a generic interval-valued function f and a generic interval-valued matrix function $\mathbf{A}$ of the interval vector $\boldsymbol{\alpha}^I$ will be denoted in equivalent form, respectively as:

$$\begin{aligned} f^I &\equiv f(\boldsymbol{\alpha}^I) \quad \Leftrightarrow \quad f(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right]; \\ \mathbf{A}^I &\equiv \mathbf{A}(\boldsymbol{\alpha}^I) \quad \Leftrightarrow \quad \mathbf{A}(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right]. \end{aligned} \tag{5a,b}$$

2.2 Generalized interval eigenvalue problem

The vibration analysis of a $n-$DOFs undamped linear discretized structure with r uncertain-but-bounded parameters leads to the so-called generalized interval eigenvalue problem:

$$\mathbf{K}(\boldsymbol{\alpha})\boldsymbol{\phi}_j(\boldsymbol{\alpha}) = \lambda_j(\boldsymbol{\alpha})\mathbf{M}(\boldsymbol{\alpha})\boldsymbol{\phi}(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right], \quad (j = 1, 2, \ldots, n) \tag{6}$$

where $\mathbf{K}^I \equiv \mathbf{K}(\boldsymbol{\alpha}^I)$ and $\mathbf{M}^I \equiv \mathbf{M}(\boldsymbol{\alpha}^I)$ are the $n \times n$ stiffness and mass matrices of the structural system which are functions of the dimensionless uncertain parameters collected into the interval vector $\boldsymbol{\alpha}^I \in \mathbb{IR}^r$; $\lambda_j(\boldsymbol{\alpha}^I) = \omega_j^2(\boldsymbol{\alpha}^I)$ is the $j-$th squared interval natural frequency and $\boldsymbol{\phi}_j(\boldsymbol{\alpha}^I)$ is the associated eigenvector.

According to the *classical interval algebra*, the interval stiffness and mass matrices satisfy the following relationships:

$$\begin{aligned} \mathbf{K}(\boldsymbol{\alpha}^I) &= \left[\underline{\mathbf{K}}, \overline{\mathbf{K}}\right] = \left\{\mathbf{K}(\boldsymbol{\alpha}) \middle| \; \underline{k}_{ij} \le k_{ij} \le \overline{k}_{ij}\right\}; \\ \mathbf{M}(\boldsymbol{\alpha}^I) &= \left[\underline{\mathbf{M}}, \overline{\mathbf{M}}\right] = \left\{\mathbf{M}(\boldsymbol{\alpha}) \middle| \; \underline{m}_{ij} \le m_{ij} \le \overline{m}_{ij}\right\} \end{aligned} \tag{7a,b}$$

where $\underline{k}_{ij} = k_{0,ij} - \Delta k_{ij}$ and $\overline{k}_{ij} = k_{0,ij} + \Delta k_{ij}$, $\underline{m}_{ij} = m_{0,ij} - \Delta m_{ij}$ and $\overline{m}_{ij} = m_{0,ij} + \Delta m_{ij}$, are the bounds of the (i, j)-th element evaluated according to Eqs. (1a,b). In the previous equations, $\left\{ \mathbf{S}(\boldsymbol{\alpha}) \middle| P(\boldsymbol{\alpha}) \right\}$ means "the set of matrices $\mathbf{S}(\boldsymbol{\alpha})$ such that the proposition $P(\boldsymbol{\alpha})$ holds". In vibration problems, $\mathbf{K} \in \mathbf{K}^I$ and $\mathbf{M} \in \mathbf{M}^I$ are symmetric positive definite matrices.

The solution of the generalized interval eigenvalue problem is a non-trivial task since it involves the evaluation of all possible eigenvalues satisfying Eq. (6) as the matrices $\mathbf{M}(\boldsymbol{\alpha}^I)$ and $\mathbf{K}(\boldsymbol{\alpha}^I)$ assume all possible values inside the intervals (7a,b). The solutions constitute a complicated region in the real number field $\mathbb{R}$. Therefore, the objective is to evaluate for each eigensolution the narrowest interval enclosing all possible eigenvalues satisfying Eq. (6), i.e.:

$$\lambda_j(\boldsymbol{\alpha}) = \omega_j^2(\boldsymbol{\alpha}) = \left[\underline{\lambda}_j, \overline{\lambda}_j \right], \qquad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}} \right] \tag{8}$$

where $\underline{\lambda}_j$ and $\overline{\lambda}_j$, $(j = 1, 2, \ldots, n)$, are the LB and UB of the $j-$th interval eigenvalue.

The eigenvectors associated with the interval eigenvalues are also affected by the uncertainties and turn out to be bounded by interval vectors $\boldsymbol{\phi}_j \in \boldsymbol{\phi}_j(\boldsymbol{\alpha}^I)$. Since the main concern for design purposes is the variation of the natural frequencies due to structural parameter fluctuations, attention is focused herein on the evaluation of the bounds of the interval eigenvalues.

If the eigenvalues are monotonic functions of the uncertain parameters $\alpha_i \in \alpha_i^I = \left[\underline{\alpha}_i, \overline{\alpha}_i \right]$, $(i = 1, 2, \ldots, r)$, then the exact values of the LB and UB, $\underline{\lambda}_j$ and $\overline{\lambda}_j$, $(j = 1, 2, \ldots, n)$, can be obtained by applying the *vertex method* [25]. Indeed, the bounds of the interval eigenvalues occur at the extreme points of the uncertain parameter vector $\boldsymbol{\alpha}^I \in \mathbb{IR}^r$. The application of the *vertex method* involves the solution of 2^r deterministic eigenvalue problems, as many as are the possible combinations of the bounds of the interval uncertainties, and the subsequent evaluation of the maximum and minimum eigenvalue for each eigensolution. Such a combinatorial procedure becomes prohibitive as the number of uncertain parameters increases. Some approximate methods available in the literature enable to efficiently replace the combinatorial procedure by the solution of two deterministic eigenvalue problems (see e.g. [7-10]). The main limitation of such methods lies in the inability to take into account dependencies between mass and stiffness uncertainties according to the actual variability of structural properties in real engineering problems.

3. PROPOSED PROCEDURE FOR EVALUATING EIGENVALUE BOUNDS

The aim of this study is to propose an efficient procedure for evaluating the bounds of the interval natural frequencies able to take into account the dependencies between uncertain parameters and their actual variability in real structural systems. For this purpose, based on their influence on the structural matrices, the uncertain parameters are subdivided into three groups: 1) uncertainties affecting only the stiffness matrix, such as the Young's moduli of the material, denoted by $\alpha^I_{K,i}$, $(i=1,2,\dots,r_K)$; 2) uncertainties affecting only the mass matrix, such as lumped masses in discretized structures or mass density, denoted by $\alpha^I_{M,i}$, $(i=r_K+1, r_K+2,\dots,r_K+r_M)$; 3) uncertainties affecting simultaneously the stiffness and mass matrices, such as the cross-sectional areas or lengths of the structural elements, denoted by $\alpha^I_{KM,i}, (i=r_K+r_M+1, r_K+r_M+2,\dots,r_K+r_M+r_{KM})$. Based on the above classification, the interval vector $\boldsymbol{\alpha}^I$ of order $r=r_K+r_M+r_{KM}$ listing the uncertain parameters can be partitioned as follows:

$$\boldsymbol{\alpha}^I = \begin{bmatrix} \boldsymbol{\alpha}^I_K & \boldsymbol{\alpha}^I_M & \boldsymbol{\alpha}^I_{KM} \end{bmatrix}^T. \tag{9}$$

Following the *improved interval analysis via EUI* [22,23], the elements of the sub-vectors $\boldsymbol{\alpha}^I_K$, $\boldsymbol{\alpha}^I_M$ and $\boldsymbol{\alpha}^I_{KM}$ can be expressed as

$$\alpha^I_{K,i} = \Delta\alpha_{K,i}\hat{e}^I_{K,i}; \quad \alpha^I_{M,i} = \Delta\alpha_{M,i}\hat{e}^I_{M,i}; \quad \alpha^I_{KM,i} = \Delta\alpha_{KM,i}\hat{e}^I_{KM,i} \tag{10a-c}$$

where $\hat{e}^I_{K,i}$, $\hat{e}^I_{M,i}$ and $\hat{e}^I_{KM,i}$ are the *EUIs* associated with the three types of uncertainties introduced above, while $\Delta\alpha_{K,i}$, $\Delta\alpha_{M,i}$ and $\Delta\alpha_{KM,i}$ are the corresponding deviation amplitudes (see Eq. (1b)). Without loss of generality, it is assumed that each of the three interval sub-vectors $\boldsymbol{\alpha}^I_K$, $\boldsymbol{\alpha}^I_M$ and $\boldsymbol{\alpha}^I_{KM}$ collects the dimensionless fluctuations of a given property in the whole structure. For instance, $\boldsymbol{\alpha}^I_K$ may be the vector listing the dimensionless fluctuations of the Young's moduli of the material in different structural elements. Obviously, in the most general case, more than three properties of a structural system may exhibit fluctuations.

If the eigenvalues are monotonic functions of the uncertain parameters, their exact bounds, $\underline{\lambda}_j$ and $\overline{\lambda}_j$, $(j=1,2,\ldots,n)$, can be evaluated by applying the *vertex method* which involves the solution of 2^r deterministic eigenvalue problems, as many as are the combinations of the bounds of the uncertain parameters. The proposed procedure avoids the onerous solution of 2^r deterministic eigenvalue problems by performing a preliminary sensitivity analysis of the eigenvalues. After some algebra, the sensitivity of the j-th eigenvalue with respect to the i-th uncertain parameter can be expressed as [1,24]:

$$s_{\lambda_j,i} = \left.\frac{\partial\lambda_j(\boldsymbol{\alpha})}{\partial\alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \boldsymbol{\phi}^T_{0,j}\mathbf{K}_i\boldsymbol{\phi}_{0,j} - \lambda_{0,j}\,\boldsymbol{\phi}^T_{0,j}\,\mathbf{M}_i\boldsymbol{\phi}_{0,j}, \quad (j=1,2,\ldots,n;\ \ i=1,2,\ldots,r). \tag{11}$$

In the previous equation, $\lambda_{0,j}$ and $\boldsymbol{\phi}_{0,j}$ are the j-th eigenvalue and eigenvector of the nominal system, solutions of the following eigenvalue problem:

$$\mathbf{K}_0\,\boldsymbol{\phi}_{0,j} = \lambda_{0,j}\,\mathbf{M}_0\,\boldsymbol{\phi}_{0,j}, \qquad (j=1,2,\ldots,n) \tag{12}$$

where

$$\mathbf{K}_0 = \mathbf{K}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}}; \quad \mathbf{M}_0 = \mathbf{M}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}} \tag{13a,b}$$

are the stiffness and mass matrices of the structure with nominal values of the uncertain parameters, i.e. $\boldsymbol{\alpha} = \mathbf{0}$. Furthermore, in Eq. (11) $\mathbf{K}_i$ and $\mathbf{M}_i$ denote $n \times n$ matrices given by:

$$\mathbf{K}_i = \left.\frac{\partial \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} ; \quad \mathbf{M}_i = \left.\frac{\partial \mathbf{M}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} , \quad (i = 1, 2, \dots, r). \tag{14a,b}$$

As known, the sensitivity defined in Eq. (11) gives information about the change of the j-th eigenvalue due to a variation of the i-th structural parameter α_i with respect to the nominal value. Specifically, within a small range around $\boldsymbol{\alpha} = \mathbf{0}$, if $s_{\lambda_j,i} > 0$, then the j-th eigenvalue is an increasing function of the parameter α_i; conversely, if $s_{\lambda_j,i} < 0$, then the j-th eigenvalue is a decreasing function of the parameter α_i. Based on the knowledge of the sensitivities $s_{\lambda_j,i}$ $(i = 1, 2, \dots, r)$, the combinations of the extreme values of the uncertain parameters corresponding to the bounds of the j-th eigenvalue can be found as follows:

$$\begin{array}{l} \text{if } s_{\lambda_j,i} > 0, \text{ then } \alpha_{j,i}^{(\mathrm{UB})} = \overline{\alpha}_i, \; \alpha_{j,i}^{(\mathrm{LB})} = \underline{\alpha}_i; \\ \text{if } s_{\lambda_j,i} < 0, \text{ then } \alpha_{j,i}^{(\mathrm{UB})} = \underline{\alpha}_i, \; \alpha_{j,i}^{(\mathrm{LB})} = \overline{\alpha}_i, \quad (j = 1, 2, \dots, n; \; i = 1, 2, \dots, r). \end{array} \tag{15a,b}$$

Taking into account the partition of the vector $\boldsymbol{\alpha}$ in Eq. (9), the previous parameters are collected into the following vectors of order r:

$$\boldsymbol{\alpha}_j^{(\mathrm{UB})} = \left[\boldsymbol{\alpha}_{K,j}^{(\mathrm{UB})} \quad \boldsymbol{\alpha}_{M,j}^{(\mathrm{UB})} \quad \boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})} \right]^T ; \quad \boldsymbol{\alpha}_j^{(\mathrm{LB})} = \left[\boldsymbol{\alpha}_{K,j}^{(\mathrm{LB})} \quad \boldsymbol{\alpha}_{M,j}^{(\mathrm{LB})} \quad \boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})} \right]^T , \quad (j = 1, 2, \dots n). \tag{16a,b}$$

Then, the bounds of the eigenvalues can be evaluated solving the following two deterministic eigenvalue problems:

$$\mathbf{K}\left(\boldsymbol{\alpha}_j^{(\mathrm{LB})}\right) \boldsymbol{\phi}_j = \underline{\lambda}_j \, \mathbf{M}\left(\boldsymbol{\alpha}_j^{(\mathrm{LB})}\right) \boldsymbol{\phi}_j ; \quad \mathbf{K}\left(\boldsymbol{\alpha}_j^{(\mathrm{UB})}\right) \boldsymbol{\phi}_j = \overline{\lambda}_j \, \mathbf{M}\left(\boldsymbol{\alpha}_j^{(\mathrm{UB})}\right) \boldsymbol{\phi}_j , \quad (j = 1, 2, \dots n). \tag{17a,b}$$

To cover the most common situations occurring in structural analysis, three different cases will be examined where the uncertain parameters affecting the mass and stiffness matrices are assumed: *i*) fully disjoint (e.g. Young's moduli and mass density); *ii*) completely coincident (e.g. cross-sectional areas or lengths); *iii*) partially coincident (e.g. Young's moduli, mass density and cross-sectional areas or lengths).

3.1 Mass and stiffness matrices affected by fully disjoint uncertainties (CASE I)

Case I concerns structural systems whose stiffness and mass matrices are affected by fully disjoint uncertain parameters. As a typical example, consider the case of structures with uncertain Young's modulus and mass density of the material.

Under this assumption, the vector (9) collecting the uncertain parameters reduces to $\boldsymbol{\alpha}^I = \begin{bmatrix} \boldsymbol{\alpha}_K^I & \boldsymbol{\alpha}_M^I \end{bmatrix}^T$, and the mass and stiffness matrices turn out to be functions of fully disjoint parameters, say $\mathbf{K}^I = \mathbf{K}(\boldsymbol{\alpha}_K^I)$ and $\mathbf{M}^I = \mathbf{M}(\boldsymbol{\alpha}_M^I)$. Furthermore, it can be readily verified that the sensitivities of the eigenvalues with respect to the stiffness parameters are given by:

$$s_{\lambda_j,i} = \left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_{K,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \boldsymbol{\phi}_{0,j}^T \mathbf{K}_i \boldsymbol{\phi}_{0,j}, \quad (j=1,2,\ldots,n;\ \ i=1,2,\ldots,r_K). \tag{18}$$

Similarly, the sensitivities of the eigenvalues with respect to the parameters affecting only the mass matrix take the following form:

$$s_{\lambda_j,i} = \left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_{M,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = -\lambda_{0,j}\, \boldsymbol{\phi}_{0,j}^T \mathbf{M}_i \boldsymbol{\phi}_{0,j} \quad (j=1,2,\ldots,n;\ \ i=r_K+1, r_K+2,\ldots,r_K+r_M). \tag{19}$$

Taking into account that the matrices $\mathbf{K}_i$ and $\mathbf{M}_i$ (see Eq. (14a,b)) are positive semi-definite and that $\boldsymbol{\phi}_{0,j}$ are the eigenvectors of the nominal structure, the combinations of the extreme values of the uncertain parameters giving the bounds of the eigenvalues can be determined as:

$$\left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_{K,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \boldsymbol{\phi}_{0,j}^T \mathbf{K}_i \boldsymbol{\phi}_{0,j} > 0 \Rightarrow \alpha_{j,i}^{(\mathrm{UB})} = \alpha_{K,j,i}^{(\mathrm{UB})} = \bar{\alpha}_{K,i}, \quad \alpha_{j,i}^{(\mathrm{LB})} = \alpha_{K,j,i}^{(\mathrm{LB})} = \underline{\alpha}_{K,i}, \quad (i = 1, 2, \ldots, r_K);$$

$$\left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_{M,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = -\lambda_{0,j}\, \boldsymbol{\phi}_{0,j}^T \mathbf{M}_i \boldsymbol{\phi}_{0,j} < 0 \Rightarrow \alpha_{j,i}^{(\mathrm{UB})} = \alpha_{M,j,i}^{(\mathrm{UB})} = \underline{\alpha}_{M,i}, \quad \alpha_{j,i}^{(\mathrm{LB})} = \alpha_{M,j,i}^{(\mathrm{LB})} = \bar{\alpha}_{M,i}, \quad (i = r_K + 1, r_K + 2, \ldots, r_K + r_M). \tag{20a,b}$$

Based on the information given by the preliminary sensitivity analysis, within a small range around $\boldsymbol{\alpha} = \mathbf{0}$, all eigenvalues turn out to be monotonic increasing and decreasing functions of the parameters $\alpha_{K,i}$ and $\alpha_{M,i}$ affecting only the stiffness and mass matrix, respectively.

Collecting the combinations of the extreme values of the uncertain parameters into the following vectors:

$$\boldsymbol{\alpha}_j^{(\mathrm{UB})} = \left[\boldsymbol{\alpha}_{K,j}^{(\mathrm{UB})} \quad \boldsymbol{\alpha}_{M,j}^{(\mathrm{UB})} \right]^T = \left[\bar{\boldsymbol{\alpha}}_K \quad \underline{\boldsymbol{\alpha}}_M \right]^T; \quad \boldsymbol{\alpha}_j^{(\mathrm{LB})} = \left[\boldsymbol{\alpha}_{K,j}^{(\mathrm{LB})} \quad \boldsymbol{\alpha}_{M,j}^{(\mathrm{LB})} \right]^T = \left[\underline{\boldsymbol{\alpha}}_K \quad \bar{\boldsymbol{\alpha}}_M \right]^T, \quad (j = 1, 2, \ldots, n), \tag{21a,b}$$

the bounds of the eigenvalues $\lambda_j(\boldsymbol{\alpha})$, $(j = 1, 2, \ldots, n)$, with $\boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\boldsymbol{\alpha}_K^I \quad \boldsymbol{\alpha}_M^I \right]^T$, can be evaluated solving the following two deterministic eigenvalue problems:

$$\mathbf{K}(\underline{\boldsymbol{\alpha}}_K)\boldsymbol{\phi}_j = \underline{\lambda}_j \mathbf{M}(\bar{\boldsymbol{\alpha}}_M)\boldsymbol{\phi}_j; \quad \mathbf{K}(\bar{\boldsymbol{\alpha}}_K)\boldsymbol{\phi}_j = \bar{\lambda}_j \mathbf{M}(\underline{\boldsymbol{\alpha}}_M)\,\boldsymbol{\phi}_j. \tag{22a,b}$$

Notice that the first eigenproblem is defined setting $\hat{e}_{K,i}^I = -1$, $(i = 1, 2, \ldots, r_K)$ and $\hat{e}_{M,i}^I = +1$, $(i = r_K + 1, r_K + 2, \ldots, r_K + r_M)$. Conversely, in the definition of the second eigenvalue problem, providing the UB of the eigenvalues, the stiffness and mass uncertainties are set simultaneously at their UB and LB, respectively, namely $\hat{e}_{K,i}^I = +1$, $(i = 1, 2, \ldots, r_K)$ and $\hat{e}_{M,i}^I = -1$, $(i = r_K + 1, r_K + 2, \ldots, r_K + r_M)$. Finally, it is observed that such combinations of the extreme values of the uncertain parameters give the bounds of the eigenvalues for all eigensolutions.

As will be shown through numerical results, the LB and UB of the eigenvalues obtained as solution of the eigenvalue problems in Eqs. (22a,b) are always coincident with those provided by the *vertex method*. Furthermore, if only stiffness uncertainties are present, the proposed approach reduces to the one developed by Modares *et al.* [10]. In this connection, the present method turns out to be more general than the procedure proposed in Ref [10] which is applicable to structures involving uncertain stiffness properties only.

3.2 Mass and stiffness matrices affected by the same uncertainties (CASE II)

Case II refers to structures with uncertain-but-bounded parameters affecting simultaneously the stiffness and mass matrices. This circumstance occurs, for instance, when the cross-sectional areas or the lengths of the structural elements are uncertain. In this case, the interval vector (9) collecting the uncertain parameters reduces to $\boldsymbol{\alpha}^I = \boldsymbol{\alpha}_{KM}^I$, and the mass and stiffness matrices are functions of the same parameters, i.e.:

$$\mathbf{K}^I = \mathbf{K}(\boldsymbol{\alpha}_{KM}^I); \quad \mathbf{M}^I = \mathbf{M}(\boldsymbol{\alpha}_{KM}^I). \tag{23a,b}$$

The eigenvalue sensitivities take the general expression in Eq. (11), i.e.:

$$\left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left.\frac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_{KM,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \boldsymbol{\phi}_{0,j}^T \mathbf{K}_i \boldsymbol{\phi}_{0,j} - \lambda_{0,j}\, \boldsymbol{\phi}_{0,j}^T \mathbf{M}_i \boldsymbol{\phi}_{0,j}, \quad (j=1,2,\ldots,n; \;\; i=1,2,\ldots,r_{KM} \equiv r). \tag{24}$$

Hence, the combinations of the extreme values of the uncertain parameters giving the bounds of the eigenvalues can be determined as:

$$\begin{cases} \text{if } \left.\dfrac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_{KM,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} > 0 \text{ then } \alpha_{j,i}^{(\mathrm{UB})} = \alpha_{KM,j,i}^{(\mathrm{UB})} = \overline{\alpha}_{KM,i}; \quad \alpha_{j,i}^{(\mathrm{LB})} = \alpha_{KM,j,i}^{(\mathrm{LB})} = \underline{\alpha}_{KM,i}; \\ \text{if } \left.\dfrac{\partial \lambda_j(\boldsymbol{\alpha})}{\partial \alpha_{KM,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} < 0 \text{ then } \alpha_{j,i}^{(\mathrm{UB})} = \alpha_{KM,j,i}^{(\mathrm{UB})} = \underline{\alpha}_{KM,i}; \quad \alpha_{j,i}^{(\mathrm{LB})} = \alpha_{KM,j,i}^{(\mathrm{LB})} = \overline{\alpha}_{KM,i}, \end{cases} \tag{25a,b}$$
$$(j=1,2,\ldots,n; \;\; i=1,2,\ldots,r_{KM} \equiv r).$$

The bounds of the eigenvalues $\lambda_j(\boldsymbol{\alpha})$, $(j=1,2,\ldots,n)$, with $\boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \boldsymbol{\alpha}_{KM}^I$, can be evaluated as solution of the following two deterministic eigenproblems:

$$\mathbf{K}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}\right)\boldsymbol{\phi}_j = \underline{\lambda}_j\,\mathbf{M}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}\right)\boldsymbol{\phi}_j\,; \quad \mathbf{K}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})}\right)\boldsymbol{\phi}_j = \overline{\lambda}_j\,\mathbf{M}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})}\right)\boldsymbol{\phi}_j\,. \tag{26a,b}$$

Notice that, according to the philosophy of the *improved interval analysis via EUI* [22,23], the above eigenvalue problems are defined assuming the same combination of the extreme values of the uncertain parameters in the evaluation of the mass and stiffness matrices, say the *EUIs* $\hat{e}_{KM,i}^{I} = [-1,+1]$ are set simultaneously at their lower or upper bounds. Specifically, for each eigensolution, the matrices $\mathbf{K}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}\right)$ and $\mathbf{K}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})}\right)$ are associated with the matrices $\mathbf{M}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}\right)$ and $\mathbf{M}\left(\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})}\right)$, respectively. It follows that Eqs. (26a,b) allow to overcome the inconsistency inherent in the EIP [9], which provides the eigenvalue bounds as solution of two deterministic eigenproblems derived setting the interval parameters $\boldsymbol{\alpha}_{KM}^{I}$ at opposite extremes when evaluating the stiffness and mass matrices, despite they always represent the same physical properties.

The proposed approach is much more efficient than the *vertex method* from a computational point of view. Indeed, to evaluate the bounds of the n eigenvalues, it requires the evaluation of $n \times r$ sensitivities (see Eqs. (24)) and the solution of $2 \times n$ eigenvalue problems (see Eqs.(26a,b)), while the *vertex method* involves the solution of 2^r eigenvalue problems.

Finally, it is worth emphasizing that Eqs. (26a,b) yield the exact bounds only if the eigenvalues are monotonic functions of the uncertain parameters $\alpha_{KM,i}$.

3.3 Mass and stiffness matrices affected by partially coincident uncertainties (CASE III)

Case III concerns the general problem of structures with partially coincident uncertain-but-bounded parameters affecting the stiffness and mass matrices. In this case, the interval vector collecting the uncertain parameters is defined as in Eq.(9), so that:

$$\mathbf{K}^{I} = \mathbf{K}(\boldsymbol{\alpha}_{K}^{I}, \boldsymbol{\alpha}_{KM}^{I})\,; \quad \mathbf{M}^{I} = \mathbf{M}(\boldsymbol{\alpha}_{M}^{I}, \boldsymbol{\alpha}_{KM}^{I}). \tag{27a,b}$$

Equations (20a,b) have shown that, due to the positive semi-definiteness of the matrices $\mathbf{K}_i$ and $\mathbf{M}_i$, within a small range around $\boldsymbol{\alpha}=\mathbf{0}$, the eigenvalues are monotonic increasing and decreasing functions of the parameters $\alpha_{K,i}$ and $\alpha_{M,i}$, respectively. Thus, the combinations of the extreme values of fully disjoint stiffness and mass uncertain parameters to be considered in the evaluation of the eigenvalue bounds are those given in Eqs. (21a,b). Similarly, the sensitivities of the eigenvalues with respect to the parameters $\alpha_{KM,i}$ defined by Eq.(24), herein rewritten for the sake of clarity, allow to determine the combinations of the extreme values of such parameters corresponding to the bounds of the eigenvalues:

$$\begin{cases} \text{if } \left.\dfrac{\partial\lambda_j(\boldsymbol{\alpha})}{\partial\alpha_{KM,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \boldsymbol{\phi}_{0,j}^T\mathbf{K}_i\boldsymbol{\phi}_{0,j} - \lambda_{0,j}\,\boldsymbol{\phi}_{0,j}^T\,\mathbf{M}_i\boldsymbol{\phi}_{0,j} > 0 \text{ then } \alpha_{j,i}^{(\mathrm{UB})} = \alpha_{KM,j,i}^{(\mathrm{UB})} = \bar{\alpha}_{KM,i}; \;\; \alpha_{j,i}^{(\mathrm{LB})} = \alpha_{KM,j,i}^{(\mathrm{LB})} = \underline{\alpha}_{KM,i}; \\ \text{if } \left.\dfrac{\partial\lambda_j(\boldsymbol{\alpha})}{\partial\alpha_{KM,i}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \boldsymbol{\phi}_{0,j}^T\mathbf{K}_i\boldsymbol{\phi}_{0,j} - \lambda_{0,j}\,\boldsymbol{\phi}_{0,j}^T\,\mathbf{M}_i\boldsymbol{\phi}_{0,j} < 0 \text{ then } \alpha_{j,i}^{(\mathrm{UB})} = \alpha_{KM,j,i}^{(\mathrm{UB})} = \underline{\alpha}_{KM,i}; \;\; \alpha_{j,i}^{(\mathrm{LB})} = \alpha_{KM,j,i}^{(\mathrm{LB})} = \bar{\alpha}_{KM,i}, \end{cases}$$

$$(j=1,2,\ldots,n; \;\; i = r_K + r_M + 1, r_K + r_M + 2, \ldots, +r_K + r_M + r_{KM} \equiv r). \tag{28a,b}$$

Based on the previous sensitivity information, the bounds of the eigenvalues $\lambda_j(\boldsymbol{\alpha})$, $(j=1,2,\ldots,n)$, with $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = \begin{bmatrix}\boldsymbol{\alpha}_K^I & \boldsymbol{\alpha}_M^I & \boldsymbol{\alpha}_{KM}^I\end{bmatrix}^T$, can be evaluated solving the following two deterministic eigenvalue problems:

$$\mathbf{K}\left(\boldsymbol{\alpha}_j^{(\mathrm{LB})}\right)\boldsymbol{\phi}_j = \underline{\lambda}_j\,\mathbf{M}\left(\boldsymbol{\alpha}_j^{(\mathrm{LB})}\right)\boldsymbol{\phi}_j\,; \quad \mathbf{K}\left(\boldsymbol{\alpha}_j^{(\mathrm{UB})}\right)\boldsymbol{\phi}_j = \bar{\lambda}_j\,\mathbf{M}\left(\boldsymbol{\alpha}_j^{(\mathrm{UB})}\right)\boldsymbol{\phi}_j \tag{29a,b}$$

where

$$\begin{aligned} \boldsymbol{\alpha}_j^{(\mathrm{UB})} &= \begin{bmatrix}\boldsymbol{\alpha}_{K,j}^{(\mathrm{UB})} & \boldsymbol{\alpha}_{M,j}^{(\mathrm{UB})} & \boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})}\end{bmatrix}^T = \begin{bmatrix}\bar{\boldsymbol{\alpha}}_K & \underline{\boldsymbol{\alpha}}_M & \boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})}\end{bmatrix}^T \\ \boldsymbol{\alpha}_j^{(\mathrm{LB})} &= \begin{bmatrix}\boldsymbol{\alpha}_{K,j}^{(\mathrm{LB})} & \boldsymbol{\alpha}_{M,j}^{(\mathrm{LB})} & \boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}\end{bmatrix}^T = \begin{bmatrix}\underline{\boldsymbol{\alpha}}_K & \bar{\boldsymbol{\alpha}}_M & \boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}\end{bmatrix}^T, \quad (j=1,2,\ldots n). \end{aligned} \tag{30a,b}$$

Notice that, according to the philosophy of the *improved interval analysis via EUI* [22,23], the *EUIs* $\hat{e}^{I}_{KM,i}$ must take the same value in the evaluation of the mass and stiffness matrices, while the *EUIs* $\hat{e}^{I}_{K,i}$ and $\hat{e}^{I}_{M,i}$ associated with fully disjoint uncertainties vary independently. Furthermore, it is observed, that the combinations of the extreme values of the uncertain parameters $\alpha_{K,i}$ and $\alpha_{M,i}$ affecting only the stiffness and mass matrices, respectively, are the same for all eigenvalues and are known a priori on account of the properties of the matrices $\mathbf{K}_i$ and $\mathbf{M}_i$. Conversely, the vectors $\boldsymbol{\alpha}^{(\mathrm{UB})}_{KM,j}$ and $\boldsymbol{\alpha}^{(\mathrm{LB})}_{KM,j}$, in general, are different for each eigensolution since their definition results from sensitivity analysis. It follows that the computational effort is the same as that required in CASE II examined in the previous section.

If $\boldsymbol{\alpha}_{KM} = \mathbf{0}$, the uncertain parameters turn out to be fully disjoint and CASE I is recovered (see Eqs. (22a,b)). Similarly, if $\boldsymbol{\alpha}_{K} = \mathbf{0}$ and $\boldsymbol{\alpha}_{M} = \mathbf{0}$, the mass and stiffness matrices are affected by the same uncertain parameters and the deterministic eigenproblems (29a,b) reduce to those obtained in CASE II (see Eqs. (26a,b)).

The efficiency of the proposed approach lies in the capability of predicting the combinations of the extreme values of the uncertain parameters corresponding to the bounds of the eigenvalues based on the information given by sensitivity analysis along with the physical meaning of uncertainties. In particular, the cases examined above have shown that eigenvalue sensitivities actually need to be computed only for uncertainties affecting simultaneously the stiffness and mass matrices, say for geometrical uncertainties. The proposed solution is exact as long as the eigenvalues are monotonic functions of the uncertain parameters.

4. NUMERICAL APPLICATIONS

Three examples concerning truss structures of different complexity and a FE modeled cantilever beam are presented. The accuracy of the proposed procedure is assessed for the three cases of fully disjoint, completely coincident and partially coincident mass and stiffness uncertainties discussed above. For validation purposes, the proposed estimates of the LB and UB of the eigenvalues are contrasted with those provided by the *vertex method*. If the eigenvalues are monotonic functions of the uncertain parameters, the proposed approach yields the same results of the *vertex method*.

In the various cases, the influence of the uncertain parameters on the eigenvalues is scrutinized by evaluating the so-called *coefficient of interval uncertainty*, C_{λ_j}, defined as the ratio between the deviation amplitude and the midpoint value, i.e.:

$$C_{\lambda_j} = \frac{\Delta\lambda_j}{\lambda_{0,j}} = \frac{\overline{\lambda}_j - \underline{\lambda}_j}{\overline{\lambda}_j + \underline{\lambda}_j}, \qquad (j = 1, 2, \ldots, n). \tag{31}$$

The *coefficient of interval uncertainty*, C_{λ_j}, provides a measure of the dispersion of the interval eigenvalues λ_j^I around their midpoint value $\lambda_{0,j}$.

4.1 Example 1: 2-bar truss structure

The first example concerns the 2-bar truss structure depicted in Figure 1. The following geometrical and mechanical properties are assumed for the nominal structure: cross-sectional areas and Young's moduli of the bars $A_{0,i} = A_0 = 5 \cdot 10^{-4}\ \mathrm{m}^2$ and $E_{0,i} = E_0 = 2.1 \cdot 10^8\ \mathrm{kN/m}^2$ $(i = 1, 2)$, respectively; nominal lengths of the bars $L_{0,1} = 2L$ and $L_{0,2} = L\sqrt{2}$ with $L = 3\ \mathrm{m}$ (see Figure 1); material mass density $\rho_0 = 7800\ \mathrm{kg/m}^3$; nominal mass lumped at node 2 $m_0 = 1000\ \mathrm{kg}$.

In view of the system simplicity, closed-form expressions of the two eigenvalues of the structure in terms of the uncertain structural parameters, $\lambda_1 = \lambda_1(\boldsymbol{\alpha})$ and $\lambda_2 = \lambda_2(\boldsymbol{\alpha})$, here omitted for conciseness, can be derived. Then, the exact LB and UB of the eigenvalues can be readily obtained as the minimum and maximum of the functions $\lambda_1 = \lambda_1(\boldsymbol{\alpha})$ and $\lambda_2 = \lambda_2(\boldsymbol{\alpha})$ under the constraint $\boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}]$. In Table 1, the proposed estimates of the *coefficient of interval* *uncertainty* of the two eigenvalues are contrasted with the exact ones provided by the optimization procedure for the 2-bar truss with different uncertain parameters covering the three cases examined in Section 3. The optimization procedure has demonstrated that, for this simple example, in all the cases listed in Table 1, the exact bounds of the eigenvalues occur for the combination of the extreme values of the uncertain parameters predicted by the proposed approach. The same results can be obtained by applying the *vertex method* which requires 2^r eigenvalue analyses.

For instance, when the Young's moduli of the two bars and the lumped mass at node 2 are modeled as intervals (CASE I), say $E_i^I = E_0(1+\Delta\alpha \hat{e}_{E,i}^I)$, ($i=1,2$), and $m_2^I = m_0(1+\Delta\alpha \hat{e}_m^I)$ with $\Delta\alpha$ denoting the dimensionless deviation amplitude common to all parameters, and $\hat{e}_{E,i}^I$ and $\hat{e}_m^I$ the associated *EUIs*, the exact eigenvalue bounds are obtained solving the eigenvalue problems in Eqs. (22a,b).

If only the cross-sectional areas of the two bars are uncertain-but-bounded, say $A_i^I = A_0(1+\Delta\alpha \hat{e}_{A,i}^I)$, ($i=1,2$), the mass and stiffness matrices are affected by the same parameters (CASE II). As shown in Figure 2, within the interval $[-0.3,+0.3]$, the eigenvalues are monotonic increasing functions of the uncertain cross-sectional areas and the condition in Eq. (25a) is satisfied. Therefore, the exact bounds are those predicted by Eqs. (26a,b) where $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})} = \underline{\boldsymbol{\alpha}}_{KM}$ and $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})} = \overline{\boldsymbol{\alpha}}_{KM}$, ($j=1,2$), namely $\hat{e}_{A,i}^I = -1$ and $\hat{e}_{A,i}^I = +1$, ($i=1,2$), respectively.

Conversely, when the lengths of the two bars are modeled as interval parameters, $L_i^I = L_{0,i}(1+\Delta\alpha \hat{e}_{L,i}^I)$, ($i=1,2$), the exact bounds of the eigenvalues are obtained as solution of the eigenvalue problems in Eqs. (26a,b) where $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})} = \overline{\boldsymbol{\alpha}}_{KM}$ and $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})} = \underline{\boldsymbol{\alpha}}_{KM}$, ($j=1,2$) namely $\hat{e}_{L,i}^I = +1$ and $\hat{e}_{L,i}^I = -1$, ($i=1,2$), respectively. Indeed, as shown in Figure 3, within the interval $[-0.3,+0.3]$, the eigenvalues are monotonic decreasing functions of the uncertain lengths of the bars and the condition in Eq.(25b) is satisfied. Indeed, it can be readily verified that the conditions $\boldsymbol{\phi}_{0,j}^T \mathbf{K}_i \boldsymbol{\phi}_{0,j} < 0$ and $\boldsymbol{\phi}_{0,j}^T \mathbf{M}_i \boldsymbol{\phi}_{0,j} > 0$ hold for any eigensolution.

Finally, in the most general case involving uncertain Young's moduli, cross-sectional areas, lengths and lumped mass, the exact eigenvalue bounds are provided by the eigenvalue problems in Eqs. (29a,b) setting: $\hat{e}_{E,i}^I = +1$, $\hat{e}_{A,i}^I = +1$, $\hat{e}_{L,i}^I = -1$, ($i = 1,2$), and $\hat{e}_m^I = -1$ to find the UB; $\hat{e}_{E,i}^I = -1$, $\hat{e}_{A,i}^I = -1$, $\hat{e}_{L,i}^I = +1$, ($i = 1,2$), and $\hat{e}_m^I = +1$ to evaluate the LB.

By inspection of Table 1, it is observed that the dispersion around the midpoint values is the same for the two eigenvalues. To demonstrate the accuracy of the proposed method, the results reported in Table 1 are obtained considering large deviation amplitudes of the uncertain parameters, say $\Delta\alpha = 0.3$, which lead to a very large dispersion of the eigenvalues unlikely to occur in engineering practice.

4.2 Example 2: 27-bar truss structure

Let us consider the 27-bar truss structure shown in Figure 4. The nominal structure is characterized by the following geometrical and mechanical properties: cross-sectional areas and Young's moduli of the bars $A_{0,i} = A_0 = 5 \cdot 10^{-4}\ \text{m}^2$ and $E_{0,i} = E_0 = 2.1 \cdot 10^8\ \text{kN/m}^2$ $(i = 1, 2, \ldots, 27)$, respectively; nominal lengths of the bars $L_{0,i}$ $(i = 1, 2, \ldots, 27)$ specified in Figure 4 where $L = 3\ \text{m}$; material mass density $\rho_0 = 7800\ \text{kg/m}^3$. Furthermore, each node possesses a nominal lumped mass $m_0 = 1000\ \text{kg}$.

4.2.1 Uncertain Young's moduli and lumped masses (CASE I)

To validate the proposed method in the case of fully disjoint mass and stiffness uncertainties, it is assumed that the Young's moduli of nine bars and the masses lumped at nine nodes are described by interval variables, say: $E_i^I = E_0(1 + \Delta\alpha\, \hat{e}_{E,i}^I)$, $(i = 16, 17, \ldots, 24)$; $m_j^I = m_0(1 + \Delta\alpha\, \hat{e}_{m,j}^I)$,

$(j=4,5,\ldots,12)$, where $\Delta\alpha$ is the dimensionless deviation amplitude common to all uncertain parameters, while $\hat{e}^I_{E,i}$ and $\hat{e}^I_{m,j}$ are the *EUIs* associated with each interval variable. According to the proposed approach, taking into account that the Young's moduli and lumped masses enter the stiffness and mass matrices, respectively, the LB and UB of the eigenvalues are evaluated as solution of the deterministic eigenvalue problems in Eqs. (22a,b) obtained setting $\hat{e}^I_{E,i}=-1$, $\hat{e}^I_{m,j}=+1$ and $\hat{e}^I_{E,i}=+1$, $\hat{e}^I_{m,j}=-1$, $(i=16,17,\ldots,24;\ j=4,5,\ldots,12)$ respectively. In Table 2, the proposed estimates of the bounds of the first five eigenvalues along with the exact ones obtained by the *vertex method* for $\Delta\alpha=0.3$ are reported. Notice that, when the stiffness and mass matrices are affected by different uncertainties, the proposed method yields the exact LB and UB of the eigenvalues even for large uncertainty levels satisfying the condition $|\Delta\alpha|<1$. It is worth emphasizing that the *vertex method* is much more time consuming than the proposed procedure since it requires the solution of 2^{18} deterministic eigenproblems.

The influence of the uncertain parameters on the first five eigenvalues can be detected from Figure 5 where the *coefficients of interval uncertainty* for three different values of the deviation amplitude $\Delta\alpha$ of the uncertain parameters are displayed. As expected, the dispersion of the eigenvalues around their midpoint value increases as larger uncertainty levels are considered and it is different for the various modes. Furthermore, it is observed that the effect of uncertainties is such that the *coefficient of interval uncertainty* of the eigenvalues is larger than the one of the uncertain parameters which implies a larger dispersion.

4.2.2 Uncertain cross-sectional areas of the diagonal bars (CASE II)

The case of mass and stiffness matrices affected by the same parameters is first examined assuming that the cross-sectional areas of the diagonal bars of the truss structure are modeled as intervals, say $A_i^I = A_0(1+\Delta\alpha\,\hat{e}_{A,i}^I)$, ($i = 16, 17\ldots, 27$), where $\Delta\alpha$ is the dimensionless deviation amplitude common to all bars and $\hat{e}_{A,i}^I$ are the corresponding *EUIs*.

Focusing the attention on the first five eigensolutions, sensitivity analysis shows that, at least within a small range around $\boldsymbol{\alpha} = \mathbf{0}$, all eigenvalues are monotonic increasing functions of the uncertain cross-sectional areas, say Eq. (25a) is always satisfied, except in the following cases: the second eigenvalue λ_2^I is a monotonic decreasing function of the parameters A_{17}^I and A_{22}^I; the third eigenvalue λ_3^I is a monotonic decreasing function of the parameters A_{16}^I and A_{23}^I. Based on this information, the bounds of the eigenvalues λ_j^I can be evaluated by solving the deterministic eigenvalue problems in Eqs.(26a,b) with the appropriate definition of the vectors $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}$ and $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})}$.

Table 3 shows that the proposed LB and UB of the first five eigenvalues for $\Delta\alpha = 0.1$ are the same as those provided by the *vertex method*. Very small differences between the proposed and the *vertex* solutions may occur for larger uncertainties which actually imply unrealistic deviations of the geometrical properties in practical engineering applications. In general, the proposed estimates of the eigenvalue bounds are different from those obtained by applying the *vertex method* when the eigenvalues are not monotonic functions of some uncertain parameters, as may happen when geometrical uncertainties are considered.

To gain further insight into the influence of the uncertain geometrical properties on the interval eigenvalues, in Figure 6, the proposed *coefficients of interval uncertainty* of the first five eigenvalues for three values of $\Delta\alpha$ are plotted. Notice that in this case the dispersion of the eigenvalues is smaller than the one of the uncertain structural parameters.

4.2.3 Uncertain lengths of the diagonal bars (CASE II)

To further validate the accuracy of the proposed method in the case of mass and stiffness matrices affected by the same uncertain parameters, the lengths of the diagonal bars are modeled as intervals, say $L_i^I = L_{0,i}(1+\Delta\alpha\,\hat{e}_{L,i}^I)$, ($i=16,17,\ldots,27$), where $\Delta\alpha$ is the dimensionless deviation amplitude common to all bars and $\hat{e}_{L,i}^I$ are the corresponding *EUIs*.

Performing a preliminary sensitivity analysis, it is readily found that the condition in (25b) is always satisfied, namely all eigenvalues are monotonic decreasing functions of the uncertain lengths. Indeed, the conditions $\boldsymbol{\phi}_{0,j}^T \mathbf{K}_i \boldsymbol{\phi}_{0,j} < 0$ and $\boldsymbol{\phi}_{0,j}^T \mathbf{M}_i \boldsymbol{\phi}_{0,j} > 0$ hold for all eigensolutions. Therefore, the bounds of the eigenvalues can be evaluated by solving the deterministic eigenvalue problems in Eqs.(26a,b) where $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})} = \overline{\boldsymbol{\alpha}}_{KM}$ and $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{UB})} = \underline{\boldsymbol{\alpha}}_{KM}$, say $\hat{e}_{L,i}^I = +1$ and $\hat{e}_{L,i}^I = -1$, $(i=16,17,\ldots,27)$, respectively, in both the mass and stiffness matrices, for all eigensolutions.

In Table 4, the LB and UB of the first five eigenvalues obtained by the proposed procedure along with those provided by the *vertex method* for $\Delta\alpha = 0.1$ are reported. It can be seen that the deterministic eigenvalue problems in Eqs.(26a,b) yield exactly the same results obtained by performing 2^{12} eigenvalue analyses according to the *vertex method*. This means that, for each eigensolution, the exact LB and UB of the eigenvalues can be obtained considering only two combinations of the uncertain lengths corresponding to $\hat{e}_{L,i}^I = +1$ and $\hat{e}_{L,i}^I = -1$, ($i=16,17,\ldots,27$). Numerical investigations, herein omitted for conciseness, have demonstrated that the same accuracy is obtained also for arbitrarily large deviation amplitude of the uncertain lengths $|\Delta\alpha| < 1$ which however are unrealistic in practical engineering problems.

The influence of the uncertain parameters on the first five eigenvalues is scrutinized by evaluating the *coefficient of interval uncertainty* which is plotted in Figure 7 for the first five modes considering three different values of the deviation amplitude $\Delta\alpha$. As expected, the uncertain lengths of the bars have a different influence on the various eigenvalues. Like in the case of uncertain cross-sectional areas of the diagonal bars, the dispersion of the interval eigenvalues around the midpoint value is smaller than that pertaining to the uncertain parameters. This means that uncertainty in the input parameters is not amplified by the structural system.

Finally, Figure 8 displays the comparison between the proposed bounds of the first two eigenvalues versus the dimensionless deviation amplitude $\Delta\alpha$ in the presence of uncertain cross-sectional areas (see Section 4.2.2) and lengths of the diagonal bars. In both cases, the proposed estimates of the LB and UB of the eigenvalues are the same as those given by the *vertex method*. Furthermore, as expected, the width of the eigenvalue regions increases with the uncertainty level. It is also observed that, although the *coefficients of interval uncertainty* are almost the same (see Figures 6 and 7), the bounds of the first two eigenvalues pertaining to the truss with uncertain cross-sectional areas and bar lengths of the diagonal bars are quite different.

4.2.4 Uncertain Young's moduli, lengths and lumped masses (CASE III)

Let us now assume that the following parameters are modeled as intervals: $E_i^I = E_0(1+\Delta\alpha\hat{e}_{E,i}^I)$, $i = 14,15,\ldots,19$; $L_j^I = L_{0,j}(1+\Delta\alpha\hat{e}_{L,j}^I)$, $j = 20,21,\ldots,25$; $m_s^I = m_0(1+\Delta\alpha\hat{e}_{m,s}^I)$, $s = 7,8,\ldots,12$ where $\Delta\alpha$ is the deviation amplitude common to all parameters and $\hat{e}_{E,i}^I$, $\hat{e}_{L,j}^I$ and $\hat{e}_{m,s}^I$ denote the associated *EUIs*. The Young's moduli and lumped masses enter only the stiffness matrix and mass matrix, respectively, while the lengths of the bars affect both the mass and stiffness of the structure. It follows that the structural matrices are functions of partially coincident uncertain parameters as in CASE III examined in Section 3.3. As already mentioned in the previous section, all eigenvalues are monotonic decreasing functions of the bar lengths, namely the condition in Eq. (25b) is satisfied. Hence, the proposed bounds of the eigenvalues are obtained as solutions of the deterministic eigenvalue problems in Eqs. (29a,b) where

$\boldsymbol{\alpha}_j^{(\mathrm{UB})} = \left[\bar{\boldsymbol{\alpha}}_K \quad \underline{\boldsymbol{\alpha}}_M \quad \underline{\boldsymbol{\alpha}}_{KM} \right]^T$ and $\boldsymbol{\alpha}_j^{(\mathrm{LB})} = \left[\underline{\boldsymbol{\alpha}}_K \quad \bar{\boldsymbol{\alpha}}_M \quad \bar{\boldsymbol{\alpha}}_{KM} \right]^T$, namely $\hat{e}_{E,i}^I = +1$, $\hat{e}_{m,s}^I = -1$, $\hat{e}_{L,j}^I = -1$ and $\hat{e}_{E,i}^I = -1$, $\hat{e}_{m,s}^I = +1$, $\hat{e}_{L,j}^I = +1$, ($i = 14, 15, \ldots, 19$; $s = 7, 8, \ldots, 12$; $j = 20, 21, \ldots, 25$), respectively.

For comparison purpose, in Table 5, the bounds of the first five eigenvalues obtained by the proposed procedure and the *vertex method* for $\Delta\alpha = 0.3$ are reported. Notice that the deterministic eigenvalue problems in Eqs. (29a,b) yield the exact bounds of the eigenvalues even when unrealistically large variation of the geometrical parameters are considered. Also in this case, the numerical effort of the combinatorial procedure involving the solution of 2^{18} eigenvalue problems can be drastically reduced by applying the proposed approach.

To scrutinize the effect of the uncertain parameters on the interval eigenvalues, in Figure 9 the *coefficients of interval uncertainty* of the first five eigenvalues for three different deviation amplitudes $\Delta\alpha$ are plotted. As in the previous cases, the dispersion of the interval eigenvalues around their midpoint value increases with the uncertainty level and for some modes it is even larger than that pertaining to the uncertain input data thus revealing an amplification of the uncertainty itself.

4.3 Example 3: FE modeled cantilever beam

The last example concerns a cantilever Euler-Bernoulli beam discretized into $n_e = 6$ FEs (see Figure 10). The beam is characterized by the following geometrical and mechanical nominal properties: length $L = L_0 = 3\ \mathrm{m}$; Young's modulus $E_0 = 30 \cdot 10^6\ \mathrm{kN/m^2}$; rectangular cross-sectional area with width $b_0 = 0.3\ \mathrm{m}$ and height $h_0 = 0.5\ \mathrm{m}$; material mass density $\rho_0 = 2500\ \mathrm{kg/m^3}$. Each Euler-Bernoulli type FE of length $L_i = 0.5\ \mathrm{m}$ has two DOFs at each node so that the discretized beam possesses $n = 12$ DOFs. The stiffness matrix and the consistent mass matrix of order $n \times n$

can be determined as explicit functions of the geometrical and mechanical properties based on the element matrices and performing standard assembly procedure.

The proposed method is applied for evaluating the bounds of the first three interval eigenvalues of the beam in the three cases of fully disjoint, completely coincident and partially disjoint uncertainties. To this aim, the following uncertain parameters of the $n_e = 6$ FEs are considered: uncertain Young's modulus and mass density, $E_i^I = E_0(1+\Delta\alpha\hat{e}_{E,i}^I)$ and $\rho_i^I = \rho_0(1+\Delta\alpha\hat{e}_{\rho,i}^I)$, (CASE I); uncertain width of the cross-sectional area $b_i^I = b_0(1+\Delta\alpha\hat{e}_{b,i}^I)$, (CASE II); uncertain Young's modulus, mass density and width of the cross-sectional area $E_i^I = E_0(1+\Delta\alpha\hat{e}_{E,i}^I)$, $\rho_i^I = \rho_0(1+\Delta\alpha\hat{e}_{\rho,i}^I)$ and $b_i^I = b_0(1+\Delta\alpha\hat{e}_{b,i}^I)$, (CASE III). Without loss of generality, all uncertain parameters are assumed to exhibit the same deviation amplitude $\Delta\alpha$.

Numerical results omitted for brevity have shown that in CASE I, where the stiffness and mass matrices are affected by fully disjoint parameters, the deterministic eigenproblems in Eqs.(22a,b) give the exact solutions for any uncertainty level satisfying the restriction $|\Delta\alpha| < 1$. Indeed, as discussed in the previous section, the eigenvalues are monotonic increasing and decreasing functions of the parameters affecting only the stiffness (Young's moduli) and mass matrix (mass densities), respectively.

In CASE II, a preliminary sensitivity analysis is needed to predict the behavior of the eigenvalues as functions of the geometrical uncertainties b_i^I, ($i = 1, 2, \ldots, n_e$). It can be verified that, within a small range around $\boldsymbol{\alpha} = \mathbf{0}$, the first three eigenvalues are monotonic increasing functions of the parameters b_i^I except in the following cases: the first eigenvalue λ_1^I is a monotonic decreasing function of the parameters b_4^I, b_5^I and b_6^I; the second and third eigenvalues λ_2^I and λ_3^I are monotonic decreasing functions of the parameters b_2^I, b_3^I and b_6^I. Then, the bounds of the eigenvalues λ_j^I can be evaluated by solving the deterministic eigenvalue problems in Eqs.(26a,b) where the vectors $\boldsymbol{\alpha}_{KM,j}^{(\mathrm{LB})}$ and $\boldsymbol{\alpha}_{KM,j}^{(UB)}$ are defined taking into account the information given by the sensitivity analysis. Table 6 shows that the LB and UB of the first three eigenvalues obtained by the proposed procedure for $\Delta\alpha = 0.1$ are the same as those provided by the *vertex method* despite the latter requires 2^6 deterministic eigenvalue analyses. It follows, that Eqs.(26a,b) yield the exact bounds of the first three eigenvalues. As in the previous examples, differences between the proposed and *vertex* solutions may occur for larger uncertainties which, however, correspond to unrealistic variations of the beam geometry.

Finally, in CASE III, the LB and UB of the first three interval eigenvalues λ_j^I can be estimated by solving the eigenvalue problems in Eqs. (29a,b) where the combinations of the extreme values of the parameters b_i^I affecting both the stiffness and mass matrices are defined as in CASE II. Since CASE III includes both the previous cases and errors may occur only in the presence of uncertain geometrical parameters b_i^I, the same accuracy of CASE II is achieved.

Figure 11 shows the comparison between the *coefficient of interval uncertainty* of the first three interval eigenvalues in the three cases examined above for a deviation amplitude of the uncertain parameters $\Delta\alpha = 0.1$. The proposed estimates are always coincident with those provided by the *vertex method*. Furthermore, it is observed that in CASE I, the uncertain parameters produce the same dispersion of the first three eigenvalues with respect to the nominal value. Conversely, in CASES II and III, uncertainties have a different influence on the first three eigenvalues.

5. CONCLUSIONS

An efficient procedure for the solution of the generalized interval eigenvalue problem arising in vibration analysis of linear undamped structures with uncertain-but-bounded parameters has been presented. The underlying idea is to properly take into account the actual variability and dependencies of uncertainties in real structural problems. This is accomplished by associating an *extra unitary interval* to each uncertain parameter according to the so-called *improved interval analysis*. The use of the *extra unitary interval* enables to handle all possible situations occurring in real structural analysis where mass and stiffness uncertainties may be fully disjoint, completely or partially coincident. In all these cases, based on the information on the monotonic behaviour of the eigenvalues with respect to each uncertain parameter given by a preliminary sensitivity analysis, the bounds of the eigenvalues have been evaluated as solution of two appropriate deterministic eigenvalue problems.

The salient features of the proposed procedure may be summarized as follows: *i*) the actual variability and dependencies of uncertainties arising in real structural problems are duly taken into account; *ii*) regardless of the type of uncertainties involved, the bounds of the eigenvalues are obtained as solution of two appropriate deterministic eigenvalue problems, thus allowing to perform the vibration analysis of real-sized structures; *iii*) as long as the eigenvalues are monotonic functions of the uncertain parameters, the proposed estimates of the bounds are exact despite no use is made of a combinatorial procedure. In particular, based on the definition of eigenvalue sensitivities it has been shown that the eigenvalues of a structural system are monotonic increasing and decreasing functions of the parameters affecting only the stiffness and mass matrices, respectively. Thus, for these types of uncertainties, the combinations of the extreme values providing the exact bounds of the eigenvalues are known a priori.

Numerical results concerning the eigenvalue analysis of two truss structures and a FE modeled cantilever beam have demonstrated the robustness and accuracy of the proposed procedure by comparison with the *vertex method*. Moreover, numerical investigations have demonstrated that the eigenvalues are significantly influenced by variations of geometrical and mechanical properties. In particular, different effects of uncertainties on the various modes have been detected.

Acknowledgements. This study has been conducted under the auspices of the research project PRIN2010-11, “Dynamic, stability and control of flexible structures”.

REFERENCES

[1] I. Elishakoff, M. Ohsaki, Optimization and Anti-Optimization of Structures under Uncertainty, Imperial College Press, London, 2010.

[2] R.E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, 1966.

[3] R.E. Moore, R.B. Kearfott, M.J. Cloud, Introduction to Interval Analysis, SIAM, Philadelphia, 2009.

[4] J. Rohn, Eigenvalues of a Symmetric interval matrix, Freiburger Intervall-Berichte 87 (1987) 67-72.

[5] A. Deif, Advanced Matrix Theory for Scientists and Engineers, Abacus Press, Tunbridge Wells, UK, 1991.

[6] Z.P. Qiu, S.H. Chen, I. Elishakoff, Natural frequencies of structures with uncertain but non-random parameters. Journal of Optimization Theory and Applications 86 (3) (1995) 669–683.

[7] I. Elishakoff (Ed.), Whys and hows in uncertainty modelling. Probability, fuzziness and anti-optimization. Springer, Vienna, 1999, 341–355.

[8] Z. Qiu, I. Elishakoff, J.H. Jr. Starnes, The bound set of possible eigenvalues of structures with uncertain but non-random parameters. Chaos, Solitons & Fractals 7(11) (1996) 1854-1857.

[9] Z.P. Qiu, X.J. Wang, M.I. Friswell, Eigenvalue bounds of structures with uncertain-but-bounded parameters. Journal of Sound and Vibration 282 (2005) 297–312.

[10] M. Modares, R.L. Mullen, R.L. Muhanna. Natural frequencies of a structure with bounded uncertainty. Journal of Engineering Mechanics (ASCE) 132 (2006) 1363-1371.

[11] W. Gao, Interval natural frequency and mode shape analysis for truss structures with interval parameters. Finite Elements in Analysis and Design 42 (2006) 471 - 477.

[12] H. Leng, Z. He, Computing eigenvalue bounds of structures with uncertain-but-non-random parameters by a method based on perturbation theory. Communications in Numerical Methods in Engineering 23 (2007) 973-982.

[13] X. Yang, S. Chen, H. Lian, Interval finite element method based on the element for eigenvalue analysis of structures with interval parameters. Structural Engineering and Mechanics 12 (6) (2001) 669-684.

[14] S.H. Chen, H. D. Lian, X. W. Yang, Interval eigenvalue analysis for structures with interval parameters. Finite Elements in Analysis and Design 39 (2003) 419-431.

[15] S.H. Chen, L. Ma, G.W. Meng, R. Guo, An efficient method for evaluating the natural frequencies of structures with uncertain-but-bounded parameters. Computers & Structures 87 (2009) 582-590.

[16] Z. Qiu, S. Chen, H. Jia, The Rayleigh quotient iteration method for computing eigenvalue bounds of structures with bounded uncertain parameters. Computers & Structures 55 (2) (1995) 221-227.

[17] H. Leng, Z. He, Q. Yuan, Computing bounds to real eigenvalues of real-interval matrices. International Journal for Numerical Methods in Engineering 74 (2008), 523-530.

[18] H. Leng, Z. He, Computation of bounds for eigenvalues of structures with interval parameters. Applied Mathematics and Computation 216 (2010) 27342739.

[19] H. Leng, Real eigenvalue bounds of standard and generalized real interval eigenvalue problems. Applied Mathematics and Computation 232 (2014) 164-171.

[20] Q. Yuan, Z. He, H. Leng, An evolution strategy method for computing eigenvalue bounds of interval matrices. Applied Mathematics and Computation 196 (2008) 257–265.

[21] Z. Zeng-Qing, C. Jian-Jun, Dynamic eigenvalue analysis of structures with interval parameters based on affine arithmetic. Structural Engineering and Mechanics 33 (2009) 539-542.

[22] G. Muscolino, A. Sofi, Stochastic analysis of structures with uncertain-but-bounded parameters via improved interval analysis. Probabilistic Engineering Mechanics 28 (2012) 152-163.

[23] G. Muscolino, A. Sofi, Bounds for the stationary stochastic response of truss structures with uncertain-but-bounded parameters. Mechanical Systems and Signal Processing 37 (2013) 163-181.

[24] A. Pownuk, General Interval FEM Program Based on Sensitivity Analysis, Proceedings f the Third International Workshop on Reliable Engineering Computing (REC 2008), Savannah (Georgia, USA), February 20-22, 2008.

[25] Z. Qiu, X. Wang, Solution theorems for the standard eigenvalue problem of structures with uncertain-but-bounded parameters. Journal of Sound and Vibration 282 (2005) 381-399.

FIGURE 1

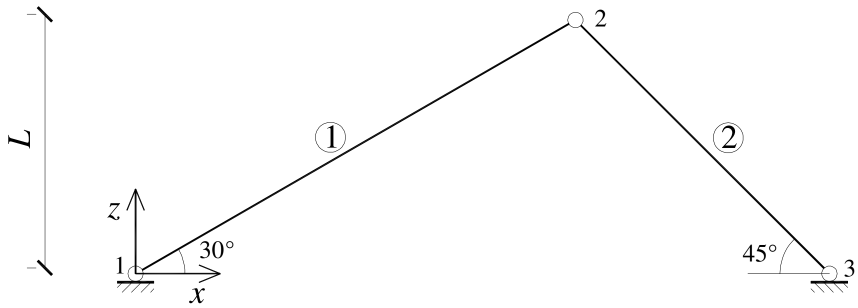


Figure 1. 2-bar truss structure with interval parameters.

Figure 2

FIGURE 2

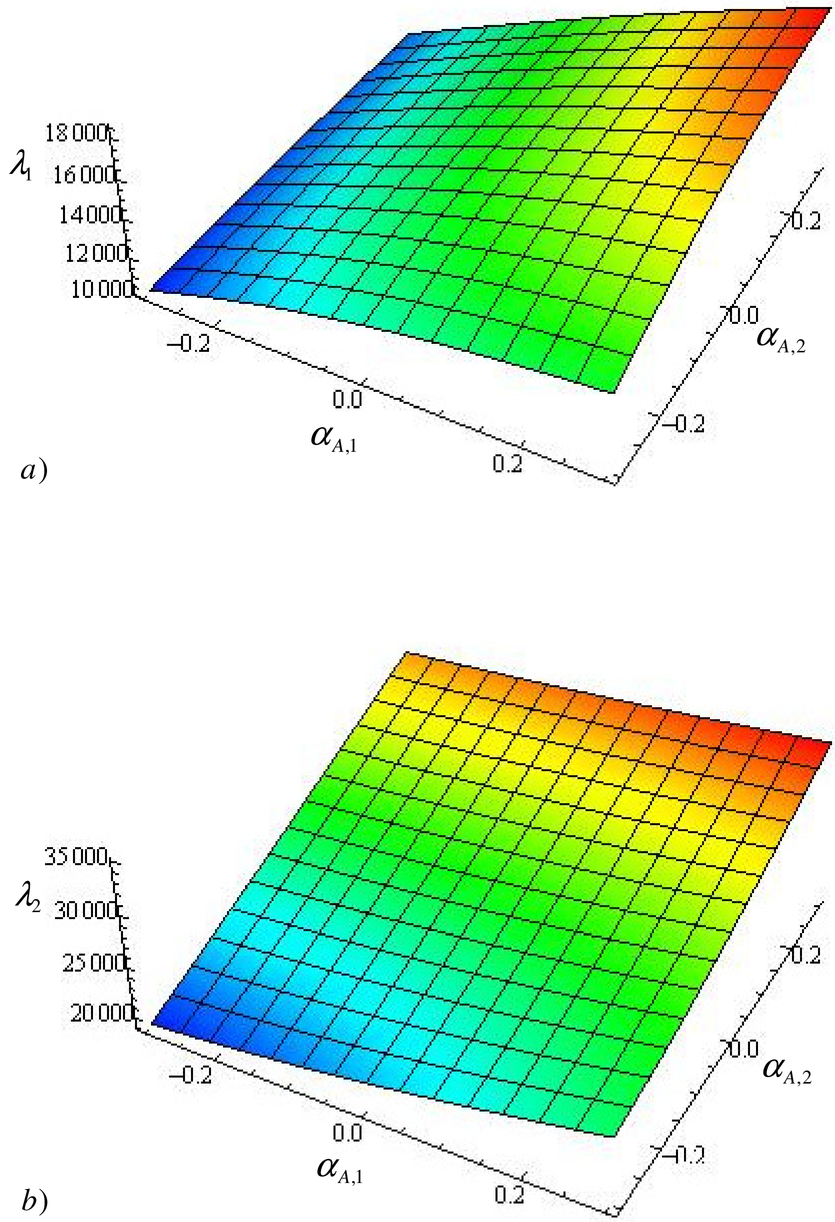


Figure 2. Plot of the (a) first and (b) second eigenvalues of the 2-bar truss versus the dimensionless fluctuations of the uncertain cross-sectional areas $\alpha_{A,1} \in \alpha_{A,1}^I = [-0.3, 0.3]$ and $\alpha_{A,2} \in \alpha_{A,2}^I = [-0.3, 0.3]$.

Figure 3

FIGURE 3

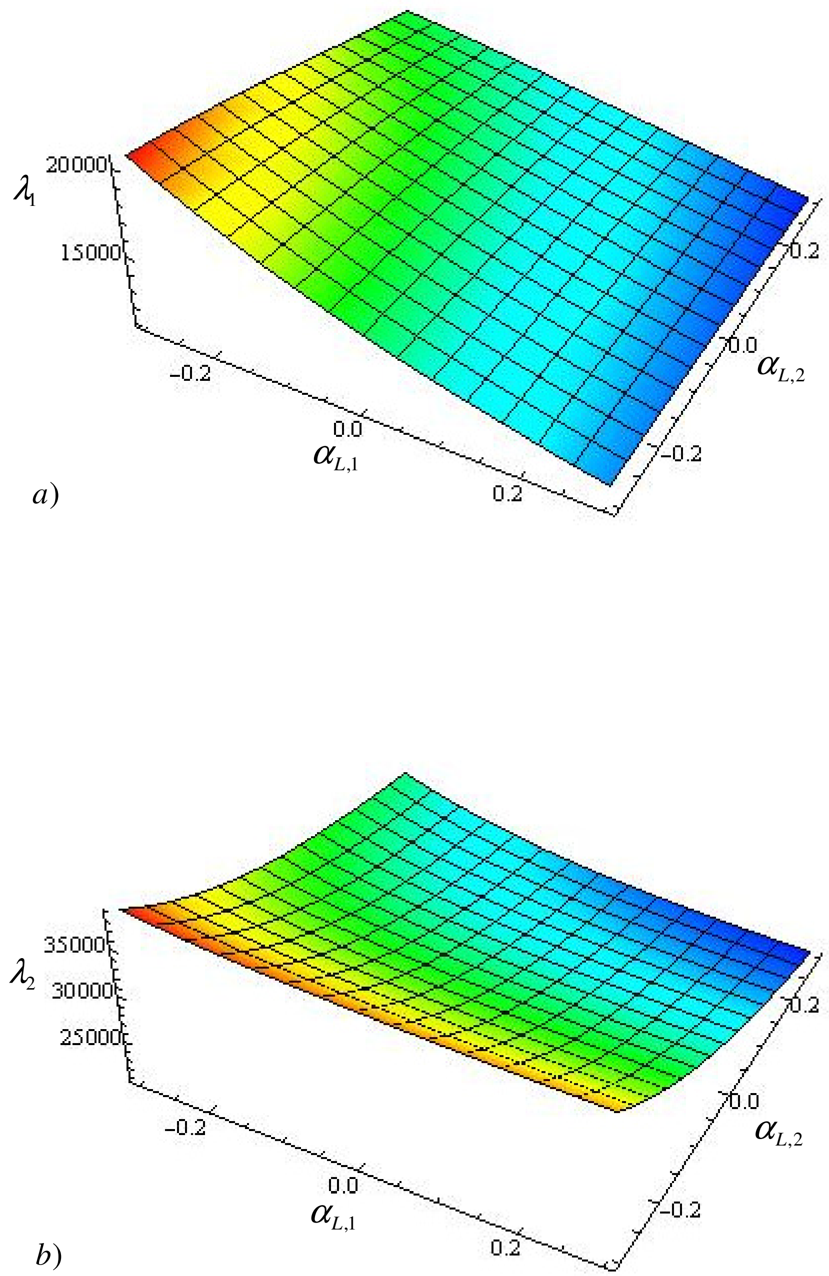


Figure 3 . Plot of the (a) first and (b) second eigenvalues of the 2-bar truss versus the dimensionless fluctuations of the uncertain lengths $\alpha_{L,1} \in \alpha_{L,1}^I = [-0.3, 0.3]$ and $\alpha_{L,2} \in \alpha_{L,2}^I = [-0.3, 0.3]$.

Figure 4

FIGURE 4

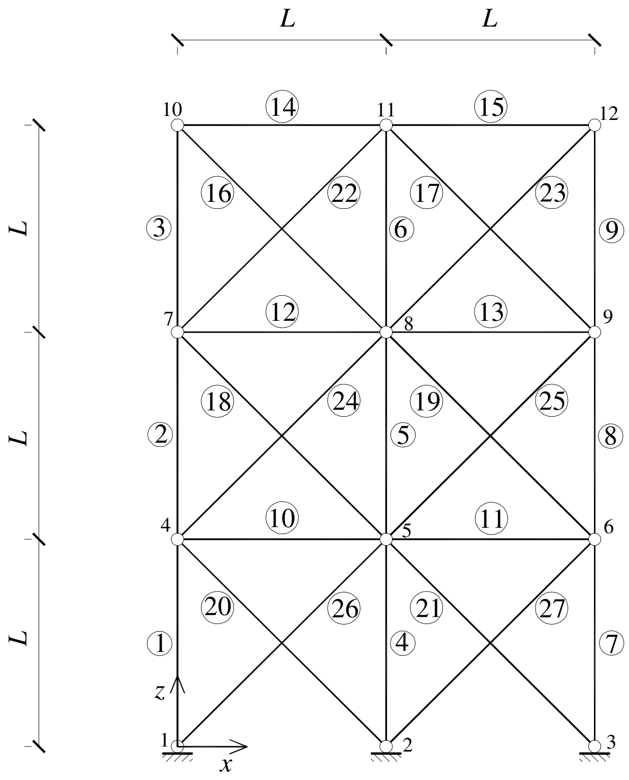


Figure 4. Truss structure with interval parameters.

Figure 5

FIGURE 5

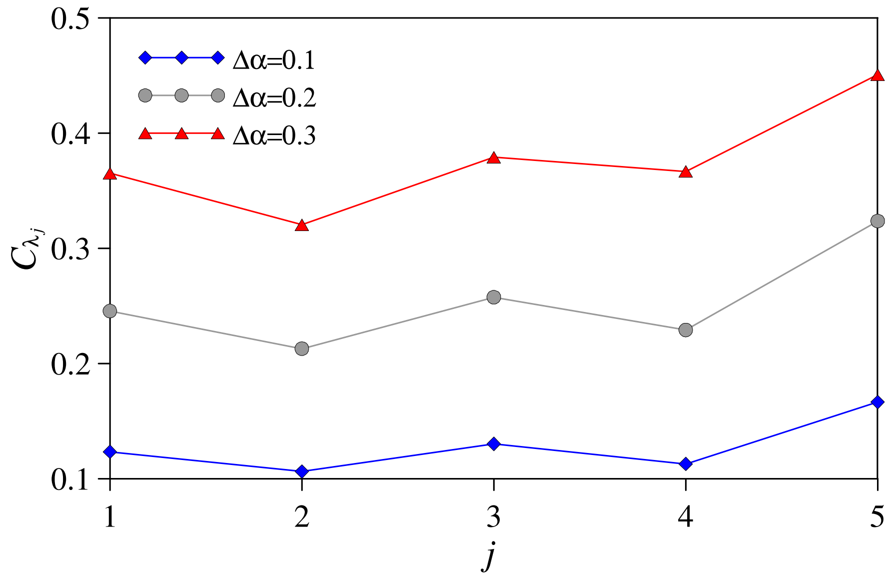


Figure 5. *Coefficient of interval uncertainty* of the first five eigenvalues of the 27-bar truss for three different deviation amplitudes of the uncertain parameters (CASE I: $E_i^I = E_0(1+\Delta\alpha\,\hat{e}_{E,i}^I)$, $i = 16, 17, \ldots, 24$; $m_j^I = m_0(1+\Delta\alpha\,\hat{e}_{m,j}^I)$, $j = 4, 5, \ldots, 12$): proposed estimates coincident with the *vertex method* solutions.

Figure 6

FIGURE 6

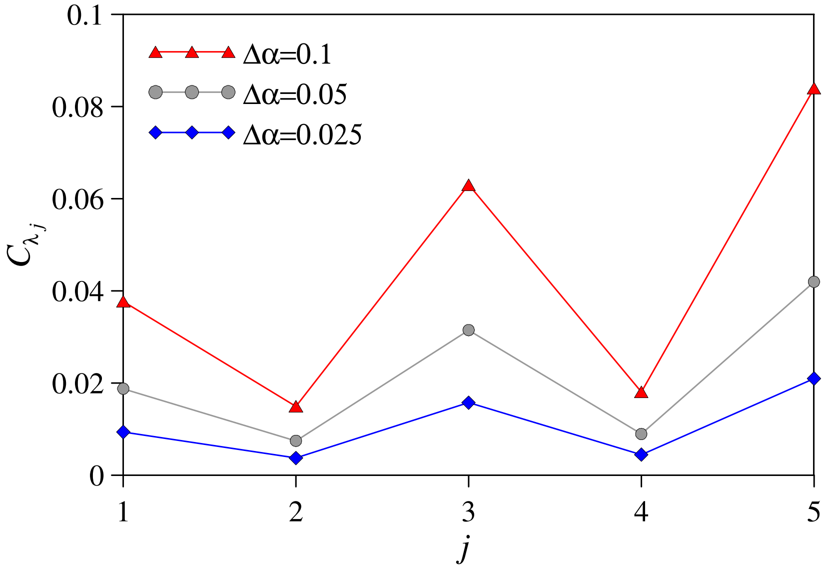


Figure 6. Proposed *coefficient of interval uncertainty* of the first five eigenvalues of the 27-bar truss for three different deviation amplitudes of the uncertain parameters (CASE II: $A_i^I = A_0(1+\Delta\alpha\hat{e}_{A,i}^I)$, $i = 16, 17, \ldots, 27$): proposed estimates coincident with the *vertex method* solutions.

Figure 7

FIGURE 7

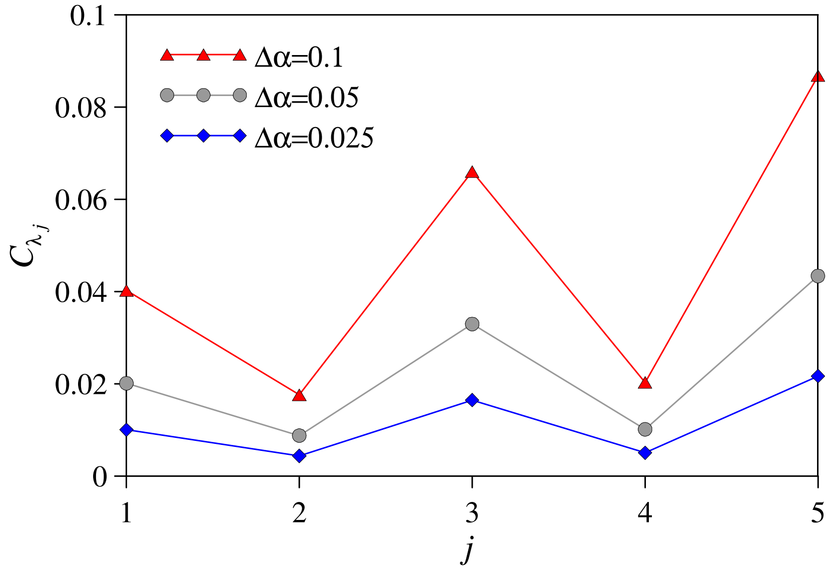
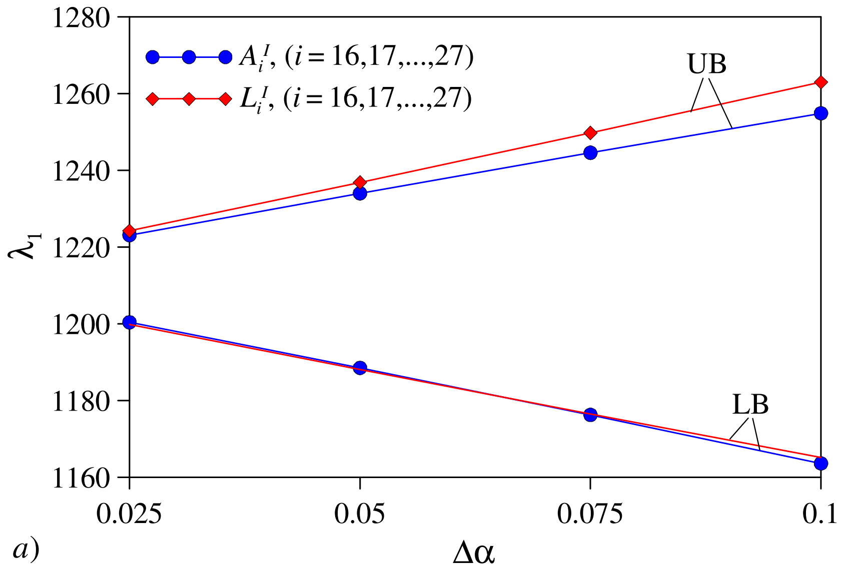


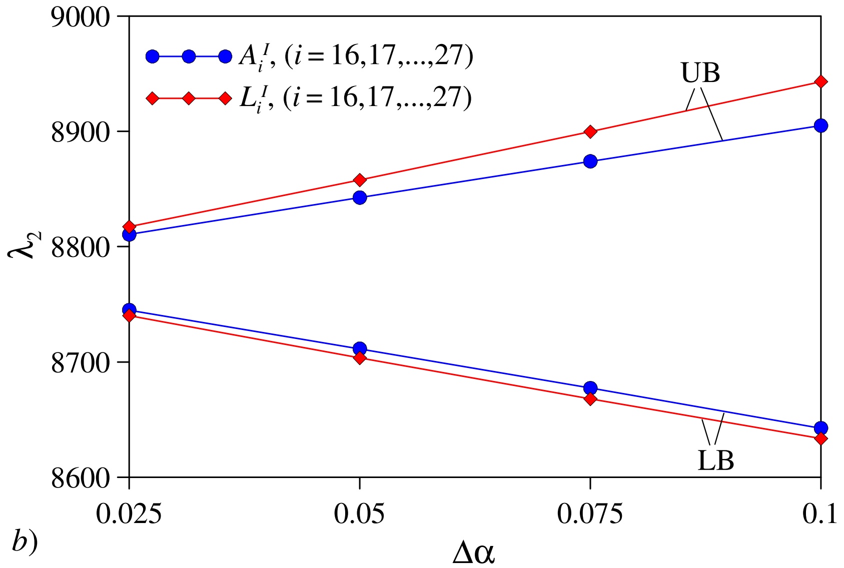
Figure 7. *Coefficient of interval uncertainty* of the first five eigenvalues of the 27-bar truss for three different deviation amplitudes of the uncertain parameters (CASE II: $L_i^I = L_{0,i}(1+\Delta\alpha\,\hat{e}_{L,i}^I)$, $i=16,17,\ldots,27$): proposed estimates coincident with the *vertex method* solutions.

Figure 8

FIGURE 8



a)



b)

Figure 8. Comparison between the (a) first and (b) second eigenvalues versus $\Delta\alpha$ for the 27-bar truss with uncertain cross-sectional areas $A_i^I = A_0(1+\Delta\alpha\hat{e}_{A,i}^I)$ and lengths $L_i^I = L_0(1+\Delta\alpha\hat{e}_{L,i}^I)$ of the diagonal bars ($i = 16,17\ldots,27$): proposed estimates coincident with the *vertex method* solutions.

Figure 9

FIGURE 9

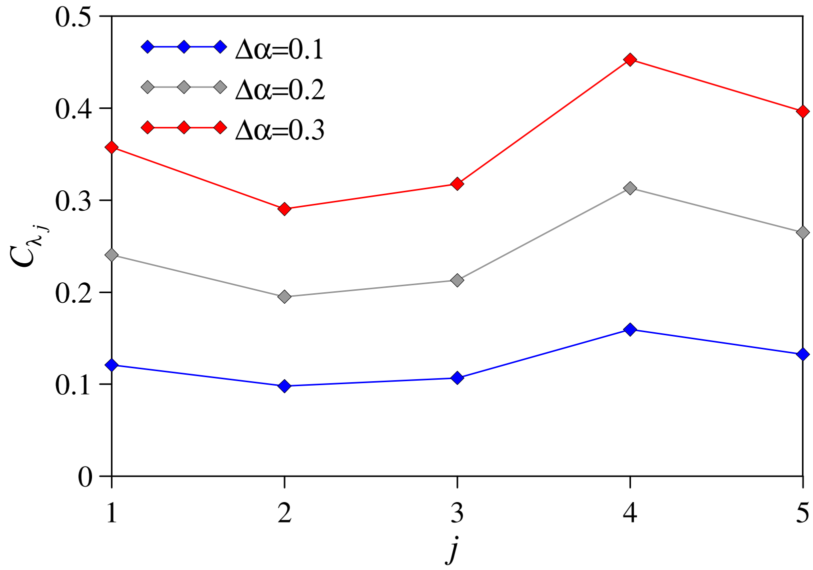


Figure 9. *Coefficient of interval uncertainty* of the first five eigenvalues of the 27-bar truss for three different deviation amplitudes of the uncertain parameters (CASE III: $E_i^I = E_0(1+\Delta\alpha \hat{e}_{E,i}^I)$, $i=14,15,\ldots,19$; $L_j^I = L_{0,j}(1+\Delta\alpha \hat{e}_{L,j}^I)$, $j=20,21,\ldots,25$; $m_s^I = m_0(1+\Delta\alpha \hat{e}_{m,s}^I)$, $s=7,9,\ldots,12$): proposed estimates coincident with the *vertex method* solutions.

Figure 10

FIGURE 10

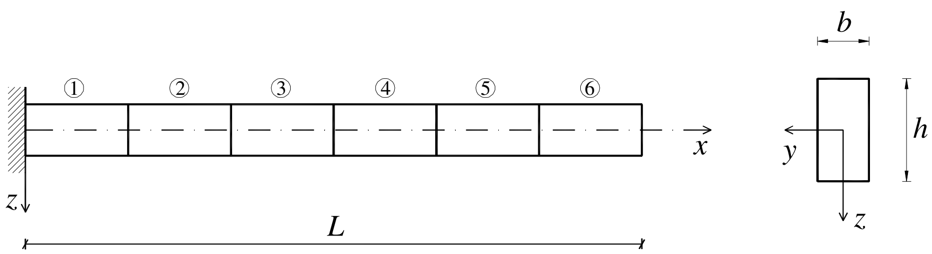


Figure 10. FE modeled cantilever beam with interval parameters.

Figure 11

FIGURE 11

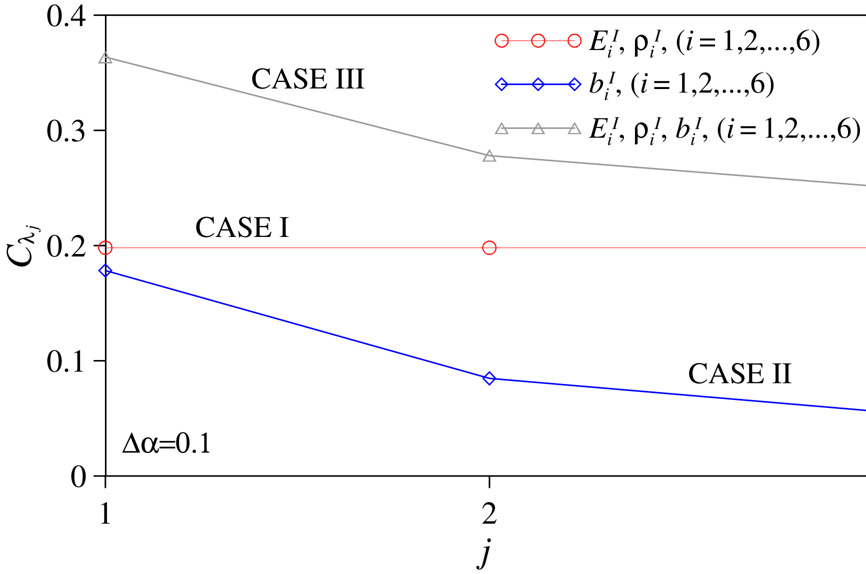


Figure 11. *Coefficient of interval uncertainty* of the first three interval eigenvalues of the FE modeled cantilever beam with fully disjoint (CASE I), fully coincident (CASE II) and partially dependent (CASE III) uncertain parameters ($\Delta\alpha = 0.1$): proposed estimates coincident with the *vertex method* solutions.

Table 1

Table. 1. Proposed and exact estimates of the *coefficient of interval uncertainty* of the eigenvalues of the 2-bar truss structure

Uncertain parameters	$\Delta\alpha = 0.3$			
	Proposed		**Exact**	
	C_{λ_1}	C_{λ_2}	C_{λ_1}	C_{λ_2}
E_1^I, E_2^I, m_2^I (CASE I)	0.5460	0.5460	0.5460	0.5460
A_1^I, A_2^I (CASE II)	0.2946	0.2946	0.2946	0.2946
L_1^I, L_2^I (CASE II)	0.3053	0.3053	0.3053	0.3053
$m_2^I; E_i^I, A_i^I, L_i^I,\ i = 1,2$ (CASE III)	0.8432	0.8432	0.8432	0.8432

Table 2

Table. 2. Proposed and *vertex* estimates of the LB and UB of the first five eigenvalues of the 27-bar truss (CASE I: $E_i^I = E_0(1+\Delta\alpha\,\hat{e}_{E,i}^I)$, $i=16,17,\ldots,24$; $m_j^I = m_0(1+\Delta\alpha\,\hat{e}_{m,j}^I)$, $j=4,5,\ldots,12$)

Mode *j*	$\Delta\alpha = 0.3$			
	Proposed		**Vertex**	
	$\underline{\lambda}_j$	$\overline{\lambda}_j$	$\underline{\lambda}_j$	$\overline{\lambda}_j$
1	850.0677	1828.0729	850.0677	1828.0729
2	6500.6504	12630.1009	6500.6504	12630.1009
3	7769.2498	17255.2954	7769.2498	17255.2954
4	21014.8905	45340.6585	21014.8905	45340.6585
5	22485.9512	59380.7113	22485.9512	59380.7113

Table 3

Table. 3. Proposed and *vertex* estimates of the LB and UB of the first five eigenvalues of the 27-bar truss (CASE II: $A_i^I = A_0(1+\Delta\alpha\,\hat{e}_{A,i}^I)$, $i=16,17,\ldots,27$)

Mode *j*	$\Delta\alpha = 0.1$			
	Proposed		**Vertex**	
	$\underline{\lambda}_j$	$\overline{\lambda}_j$	$\underline{\lambda}_j$	$\overline{\lambda}_j$
1	1163.6119	1254.8756	1163.6119	1254.8756
2	8642.5472	8905.0102	8642.5472	8905.0102
3	10440.5503	11846.0027	10440.5503	11846.0027
4	30399.0633	31518.8968	30399.0633	31518.8968
5	32241.0597	38153.2708	32241.0597	38153.2708

Table 4

Table 4. Proposed and *vertex* estimates of the LB and UB of the first five eigenvalues of the 27-bar truss (CASE II: $L_i^I = L_{0,i}(1+\Delta\alpha\,\hat{e}_{L,i}^I)$, $i=16,17,\ldots,27$)

Mode *j*	$\Delta\alpha = 0.1$			
	Proposed		**Vertex**	
	$\underline{\lambda}_j$	$\overline{\lambda}_j$	$\underline{\lambda}_j$	$\overline{\lambda}_j$
1	1165.2002	1262.9742	1165.2002	1262.9742
2	8633.4062	8943.1691	8633.4062	8943.1691
3	10474.4983	11955.0812	10474.4983	11955.0812
4	30388.5339	31647.8782	30388.5339	31647.8782
5	32425.7685	38591.2873	32425.7685	38591.2873

Table 5

Table 5. Proposed and *vertex* estimates of the LB and UB of the first five eigenvalues of the 27-bar truss (CASE III: $E_i^I = E_{0,i}(1+\Delta\alpha\hat{e}_{E,i}^I)$, $i = 14,15,\ldots,19$; $L_j^I = L_{0,j}(1+\Delta\alpha\hat{e}_{L,j}^I)$, $j = 20,21\ldots,25$; $m_s^I = m_{0,s}(1+\Delta\alpha\hat{e}_{m,s}^I)$, $s = 7,8,\ldots,12$)

Mode *j*	$\Delta\alpha = 0.3$			
	Proposed		**Vertex**	
	$\underline{\lambda}_j$	$\overline{\lambda}_j$	$\underline{\lambda}_j$	$\overline{\lambda}_j$
1	866.4222	1830.5995	866.4222	1830.5995
2	6707.3286	12198.6704	6707.3286	12198.6704
3	8280.3216	15990.0442	8280.3216	15990.0442
4	18816.8552	49945.9058	18816.8552	49945.9058
5	23173.8337	53612.4145	23173.8337	53612.4145

Table 6

Table. 6. Proposed and *vertex* estimates of the LB and UB of the first three eigenvalues of the FE modelled cantilever beam (CASE II: $b_i^I = b_0(1+\Delta\alpha\, \hat{e}_{b,i}^I)$, $i = 1, 2, \ldots, 6$)

Mode *j*	$\Delta\alpha = 0.1$			
	Proposed		**Vertex**	
	$\underline{\lambda}_j$	$\overline{\lambda}_j$	$\underline{\lambda}_j$	$\overline{\lambda}_j$
1	31807.1793	45609.6314	31807.1793	45609.6314
2	1372113.3990	1625864.4583	1372113.3990	1625864.4583
3	11184143.9897	12520595.7158	11184143.9897	12520595.7158