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Classes of Ephemeral Continua

Gianfranco Capriz and Pasquale Giovine

Abstract The qualifier ‘ephemeral’ was proposed for continuous models of bodies, such as gases, for which the generally tacit axiom of permanence of material elements fails to apply. Consequently, to their scrutiny, a Eulerian (local) approach is mandatory, such as one adopted, e.g., in molecular dynamics. Within the scheme of ephemeral continua we discuss here three essential subclasses of bodies: (1) those undergoing energy-preserving processes (in this sense hyperelastic), (2) hypoelastic bodies inspired by a type proposed by C. Truesdell and (3) a number of minor ones. We re-examine the essential issues of the general format focusing on the proposal of appropriate concepts of strains and strainings.

1 Introduction

In recent years some papers were published with the aim of formulating a dynamics of continua for which the largely tacit primal axiom of absolute persistence of material elements fails [4, 7]. The glaring instance of such a body is a gas though the occasional placing of its dynamics within a course of lectures as a chapter of standard continuum mechanics seems to imply the opposite. We quote for all from [24] page 3: “The kinetic gas .... is a continuous medium ... Not only is the kinetic theory a field theory, but also the kinetic gas is endowed with the very same field descriptors as continuum mechanics ... ” and in a footnote ... “For an elementary introduction to the basic concepts and assumptions of continuum mechanics the reader may consult the book of C.Truesdell, A First Course in Rational Mechanics ...” [23]. There the mentioned primal axiom is glossed over (page 36) by a seemingly innocent Axiom of Impenetrability: “In continuum mechanics, contrarily, the mappings $\chi(\cdot,t) : \mathscr{B} \to \chi_\Omega(\mathscr{B},t)$ is assumed bijective”, professed already also in [26], p. 244.

We labelled ‘largely tacit’ the axiom of persistence; in fact it is spelled out for ‘particles’ as a basic principle in [26], p. 244, but without a warning that, for important classes of bodies, physical circumstances deny the existence of ‘particles’. The word ‘gas’ does not appear in the Subject Index of [25]. The physical concept of ‘fluid’ is declared ‘vague’ in §32 of [25] and a definition of ‘simple fluid’ is waived to a constitutive rule. The impasse is met when, in a general approach, resort is imperative to the ‘principle of material frame-indifference’ (§19 of [25]), a principle which is spelled out only for ‘simple bodies’ made up of particles.

Recently the imposition of frame-indifference had met renewed interest in sub-grid scale computer modelling of turbulence (the literature is vast; for brevity, we call attention on [19], available on the web).

We have called ephemeral continuum the model of bodies where the axiom of persistence fails. Necessarily the ensuing theory is local in essence, and the image one envisages is of physical space divided into small cubic boxes, as broached in molecular dynamics, representative volume elements (r.v.e.) or *‘loculi’*, of edge δ; each box is envisioned as inhabited at each instant by a multitude of sundry molecules so numerous and moving so randomly to be treated, in all, as a grand canonical ensemble. In essence the theory is based on spatial averaging, but without the pretence of matching the parallel, profoundly studied, theory based on time averaging, rather with the primary goal mentioned above.

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Essentially it was proposed to substitute, within the circumstances, the trajectories of the missing material elements with wind streak lines obtained, by integration, from the wind velocity filtered over a loculus $\mathfrak{c}(x)$ (which is portrayed in its own separate space $\mathscr{E}_x$), whose mass center x is in a region $\mathscr{B}_\tau$ of the three-dimensional Euclidean space $\mathscr{E}$ occupied by the body at a time τ. The lack of identification, unique and permanent in time, of a peculiar material element placed in $\mathfrak{c}(x)$, which here is open with respect to the exchange of mass, made preferable the use of 'loculus' instead of r.v.e., usually closed with respect to it. Although, obviously, both the whole body and the entourage of each place occupied by it be observed, if at a different scale, within the same physical space, in the model they are imagined, we repeat, to belong the first to a Euclidean space $\mathscr{E}$ and each one of the others to their own Euclidean space $\mathscr{E}_x$.

To help the reader, oppressed by the forest of indispensable symbols, we use different types: lower case greek for scalars, lowercase roman for vectors, capital roman for second order tensors, bold lower case for third order tensors, capital bold for fourth order tensors

As we have mentioned, a sharp view is presumed to allow us to distinguish sub–places y within $\mathfrak{c}(x)$. We may then imagine further that, by an even sharper view, we could explore a neighborhood of each y, as one does in standard kinetic theory, but now at one remove down. That exploration should let us measure the *velocity* w of each molecule in each neighborhood and determine their *distribution* $\Theta(\tau,x;y,w)$ within the vectorial space $\mathscr{V}$ of velocities.

Remark that, as in the kinetic theory but now within the neighborhood of y, we disregard knowledge of the sub–place of each molecule, but keep note of its velocity w, to decide finally about the frequency Θ.

Actually, reference to Θ is mainly of introductory value here and would only be relevant to establish, eventually later, possible links with thermodynamics via an adapted Boltzmann equation. Here, one needs only refer to *number density* θ at y, namely

$$\theta(\tau,x;y) = \int_{\mathscr{V}} \Theta(\tau,x;y,w)\,dw \tag{1}$$

and to the *average velocity* w_* at y

$$w_*(\tau,x;y) = \theta^{-1}\int_{\mathscr{V}} w\,\Theta(\tau,x;y,w)\,dw; \tag{2}$$

we may assume that our exploration around y has given us them directly.

Thus, in particular, the *numerosity* $\omega(\tau,x)$ of all molecules in $\mathfrak{c}(x)$ at time τ would be

$$\omega(\tau,x) = \int_{\mathfrak{c}(x)} \theta(\tau,x;y)\,dy \tag{3}$$

and we are led to the first step in the invention of a continuum to be associated with the molecular (or the minutely and sparsely granular) cluster of specks, each of mass μ. The continuum would have a *mass density*

$$\rho(\tau,x) = \mu\,\delta^{-3}\,\omega(\tau,x). \tag{4}$$

For the same time and place, the obvious choice for a *gross velocity* v is the filtered wind velocity:

$$v(\tau,x) = \delta^{-3}\int_{\mathfrak{c}(x)} w_*\theta(\tau,x;y)\,dy. \tag{5}$$

For instance, the field of filtered wind velocity $v(\tau,x)$ leads directly to the field of its gradient $L(\tau,x)$ and, from the latter, one can obtain, by integration along the wind streak lines a fictitious *placement gradient* $F(\tau,x)$ via the equality

$$\dot{F} = LF; \tag{6}$$

the time-derivatives appearing above are meant to be evaluated along the wind streak-lines obtained artificially by formal integration of the equation

$$\frac{dx}{d\tau} = v(\tau,x). \tag{7}$$

But the introduction, in analogy with L, of a tensor B characterizing an affine motion best fitting locally the disorderly peculiar motion of molecules offers the opportunity of portraying local vicissitudes better than F. In particular the *suffusion*

$$\sigma := \mathrm{tr}(L-B) \tag{8}$$

puts in evidence, when multiplied by ρ, the instantaneous rate of mass change in the invented wind element. Moreover, when pursuing analogies with the standard theory, we could introduce a *mesoscopic displacement gradient* in two ways: either as the double vector G by mimicking strictly the standard case via integration of

$$\dot{G} = BG, \tag{9}$$

or by using a distinct differential equation for a double vector G_* which includes the effects of mass gain or loss $\dot{G}_* = BG_* - \frac{1}{2}\sigma G_*$. The alternative choice call for quite different stipulation of the Piola time-derivatives of vectors and tensors; we follow here the first choice opposite of that followed in earlier papers [4, 7], as details of analysis here become simpler.

The best fit mentioned above as to the prerogative of B is based on the notion of two mesoscopic tensors (per unit mass): the *tensor of Euler's inertia* Y and the *tensor moment of momentum* K

$$Y(\tau,x) := \omega^{-1}\int_{\mathfrak{c}(x)} \theta(\tau,x;y)\, y\otimes y\, dy \quad \text{and} \quad K(\tau,x) := \omega^{-1}\int_{\mathfrak{c}(x)} \theta(\tau,x;y)\, y\otimes c_*(\tau,x;y)\, dy, \tag{10}$$

respectively, where c_* is the peculiar velocities of agitation within each loculus:

$$c_*(\tau,x;y) := w_*(\tau,x;y) - v(\tau,x). \tag{11}$$

They are bound together as in the theories of quasi-rigid bodies

$$K = YB^T, \tag{12}$$

thus it is B that descends from Y and K and satisfies the (usually named) balance of moment of inertia

$$\frac{\partial Y}{\partial \tau} + (\operatorname{grad} Y)v = BY - YB^T, \tag{13}$$

which instead it is a kinematic identity due to the relation of Y with the measures of meso-strain (see comments on page 164 of [6] and equation (29) of [7]).

A third tensor has an essential rôle in the theory: the symmetric positive semi-definite *Reynolds' tensor*

$$H(\tau,x) := \omega^{-1}\int_{\mathfrak{c}(x)} \theta(\tau,x;y)\, c\otimes c\, dy, \tag{14}$$

based on the residual *peculiar velocities* c, beyond the filtered and affine components, velocities which are observer-independent:

$$c = w_* - v - By. \tag{15}$$

As explained in page 362 of [8], H is of an ambiguous nature: kinetic as a measure of collisions among molecules; dynamic as a tensorial pressure ensuing from collisions. We refer readers to [8] for more specific information. We quote only the set of equations (32)–(36) of [7] on which the theory rests: balance of mass, of momentum, of moment of momentum, of Reynolds' tensor:

– conservation of mass:

$$\frac{\partial \rho}{\partial \tau} + \operatorname{div}(\rho v) = 0; \tag{16}$$

– balance of momentum

$$\rho\left[\frac{\partial v}{\partial \tau} + (\operatorname{grad} v)\, v\right] = \rho b + \operatorname{div} T; \tag{17}$$

– balance of moment of momentum

$$\rho\left[\frac{\partial K}{\partial \tau} + (\operatorname{grad} K)v - BK - KB^T - H\right] = \rho O - A + \operatorname{div}\mathbf{m} \tag{18}$$

or, by relations (12) and (13),

$$\rho\left\{Y\left[\frac{\partial B}{\partial \tau} + (\operatorname{grad} B)v\right]^T - H\right\} = \rho O - A + \operatorname{div}\mathbf{m}; \tag{19}$$

– balance of agitation

$$\rho\left[\frac{\partial H}{\partial\tau}+(\operatorname{grad}H)v\right]=\rho J-Z+\operatorname{div}\mathbf{j}. \tag{20}$$

Again the time-derivatives appearing above are meant to be evaluated along the wind streak-lines obtained artificially by formal integration of the equation (7), but the complex expressions of the left-hand sides are consequences of the double kinetic circumstances envisaged.

As for the right–hand sides, ρb measures density of standard body forces and T is the Cauchy's tensor in (17). ρO and the third–order tensor $\mathbf{m}$ express in (18), respectively, bulk and contact external compulsions, whereas A is intimate bulk self compulsion. Formally, the right-hand side of (18) is identical to one accepted for bodies with affine microstructure and for multipolar continua [16]. In (20) ρJ, Z and third–order tensor $\mathbf{j}$ account, respectively, for external or intimate bulk compulsions and contact compulsion (see, *e.g.*, the closure equations for eddy viscosity models in the dynamics of fluids [1, 14, 17], other than, in general, [4, 7]).

Similarity will be recognized with certain renditions for pseudo–rigid (or affine) bodies [10], multipolar continua [16] and micromorphic bodies [13] (except here the attendant variability of mass measured in the terms containing σ and the added effect of agitation) or akin to the set proposed for extended thermodynamics. The version, better attuned for our purposes, is, for instance, at page 117 of [20]. The balance of moment of inertia is not inserted, as it turns out to be an identity once Y is expressed in terms of G (see equation (3.7) of [8]).

Some ancillary comments seem to us appropriate to support uncommon features of the system (16)-(20).

The first comment is, apparently, purely nominal, *i.e.*, it involves terminology perhaps now disused: the classification of quantities as extensive or intensive. It is reflected here through the different type of time-derivative appropriate to each. Y and K are extensive and Oldroyd's is the fitting version for them [21], marked here by a superimposed little circle:

$$\mathring{K}:=\frac{\partial K}{\partial\tau}+(\operatorname{grad}K)v-BK-KB^T. \tag{21}$$

If K is substituted by YB^T, in the ensuing equation (19) for B (an intensive variable), the time-rate is the standard one. H has an ambiguous valence. Here we insist on it being a purely kinetic quantity (hence intensive) and the left-hand side of (20) reflects that choice. Later, as we will recall, within the tensorial version of the kinetic energy theorem, time-changes in ρH are affected also by terms deriving from the presence of H in the left-hand side of (18), thus forming an Oldroyd's derivative as due to a contribution to time-changes in kinetic energy (an extensive quantity).

The second comment concerns the cast of the right-hand sides. The special blend proposed could be disputed, of course, but we follow the recommendations we read in most introductions to theories of complex bodies: the attributes of each particular sub-class of materials must be sufficiently reflected in the single selection of the appropriate constitutive laws.

The initial goal for offering the set above was principally to avoid certain apparent inconsistencies in gas dynamics (bound to the differently attributed status of the requirement of frame indifference), which had led to controversies (see [12, 27, 11, 15, 28]). It is appropriate to insist also on a crucial feature that distinguishes the system (16)-(20) from kindred offers: the presence of the term ρH in (18) [and (19)]. Even if the discrepancy between F and G (or, better, between L and B) were annulled by the introduction of a constraint, we would not be led to a standard theory.

The realm of the system is wide; for instance, the set above leads easily to the Navier–Stokes α-β version of the evolution equation for fluids [5]; it also leads to the equations for hypoelastic continua as introduced by Truesdell [22].

Here we explore mainly two theoretical cases. The first is fully conservative with T, A and $\mathbf{m}$ deriving from an internal energy not unlike standard cases; the jaundiced further hypothesis is a similar presumption for Z and $\mathbf{j}$ via appropriate expressions of power; the matter is admittedly inventive but could offer the grounding for some real incident. The second is based on choices for Z and $\mathbf{j}$ which are inspired by a suggestion of Truesdell [22] for the materials he called hypoelastic and, in some sense, generalizes that suggestion.

2 Strain

Before we proceed on specifics we must report comments and results already set forth in the paper [4] (see, also, [25, 26]).

The likely inexistence of a natural reference, from which to count F and G unequivocally, does not deny necessarily value, or at least service, for allied concepts of strain. Indeed, a word such as 'volume' enters even in the fundamental law of elementary gas dynamics and, in general, if it is only agreed that the reference, as already suggested (and often feasible), be isotropic, a common bare scalar factor in F and G would again attend a change.

Hence the present, brief section to recall a few definitions and reflections from elsewhere, beginning with the primary notation regarding the *right Cauchy-Green's deformation tensor* C and the corresponding one N for the meso placement together with the tensor X, which measures the *derangement* between the local macro and meso placement gradients, and the referential gradient of N:

$$C = F^T F, \quad N = G^T G, \quad X = G^{-1} F, \qquad (22)$$
$$\mathbf{c} = \mathrm{Grad}\, C = (\mathrm{grad}\, C)F, \quad \mathbf{n} = \mathrm{Grad}\, N = (\mathrm{grad}\, N)F, \quad \mathbf{C} = \mathrm{Grad}\, \mathbf{c},$$

leading to the polar multiplicative decompositions

$$F = R\, C^{\frac{1}{2}}, \quad G = R'\, N^{\frac{1}{2}}, \quad X = N^{-\frac{1}{2}} R'^T R C^{\frac{1}{2}}, \qquad (23)$$

with R and R' appropriate orthogonal tensors.

The last of relations (23) shows that X is determined by C and N only to within the orthogonal tensor Q:

$$Q = R'^T R. \qquad (24)$$

Finally, if we also introduce the vector q [q'] of the rotation associated with R [R'], which is the vector that has the direction of the axis of rotation, the appropriate orientation and the modulus equal to the angle of rotation, and for which $\dot{q}$ [$\dot{q}'$] results to be the axial vector associated to the skew tensor $\left(\dot{R}R^T\right)$ $\left[\left(\dot{R}'R'^T\right)\right]$, we can define its referential gradient tensors signified by a capital G

$$W = \mathrm{Grad}\, q = (\mathrm{grad}\; q)F, \quad \mathbf{q} = \mathrm{Grad}\, W \quad \left[W' = \mathrm{Grad}\, q' = (\mathrm{grad}\; q')F\right]. \qquad (25)$$

Conclusively, C, N, Q and W provide adequate first-order strain measures; $\mathbf{c}$, $\mathbf{n}$, $\mathbf{q}$ and their gradients could offer higher order strains. Notice that C and N are symmetric, whereas $\mathbf{c}$ and $\mathbf{n}$ are semisymmetric, *i.e.* symmetric with respect to the first two indices.

In contrast to all said above and with the intent (in the search for response functions, say) to turn attention only and ever to current events, one could call upon, again as hinted elsewhere, to local metrics depicted by the *left Cauchy-Green's deformation tensors* $\tilde{C}$ and $\tilde{N}$ and the *local derangement* $\tilde{X}$:

$$\tilde{C} = F F^T, \quad \tilde{N} = G G^T, \quad \tilde{X} = F G^{-1} \qquad (26)$$

with multiplicative decompositions

$$F = \tilde{C}^{\frac{1}{2}} R, \quad G = \tilde{N}^{\frac{1}{2}} R', \qquad (27)$$

to the rotation

$$\tilde{Q} = R R'^T \qquad (28)$$

and the *wryness* of underlying reference

$$\mathbf{w}_{ijk} = F^{-1}_{Aj}\, G_{iA,k}; \qquad (29)$$

the latter leading to *torsion* (antisymmetric in the last two indices)

$$\mathbf{h} = \frac{1}{2}(\mathbf{w} - \mathbf{w}^t) \qquad (30)$$

(where the exponent t on the top right indicates transposition of the last two indices) and to the second–order tensor measuring the *dislocation density*

$$\mathbf{e}_{iab}\, \mathbf{h}_{jab}, \qquad (31)$$

with $\mathbf{e}$ the *Ricci's alternating third–order tensor*, and the corresponding *Burgers vector* $\mathfrak{b}$ relative to any plane of normal $\hat{n}$

$$\mathfrak{b} = \left(\mathbf{e}\,\mathbf{h}^T\right) \hat{n}\,. \qquad (32)$$

When a prerequisite reference placement is available, then C, N and Q (perhaps grad N) become paramount variables. Also, instead of $\mathbf{w}$ one brings to bear $\mathbf{w}^*$:

$$\mathbf{w}^*_{NMP} = \mathbf{w}_{ijk} F_{iN} F_{jM} G_{kP} = F_{iN} G_{iM,P}, \qquad (33)$$

and its skew component in the last two indices

$$\frac{1}{2}\mathbf{e}_{MPR}U_{RN}, \tag{34}$$

with a kind of density of dislocations U:

$$U_{AB} = \frac{1}{2}\mathbf{e}_{AMP}(\mathbf{w}^*_{BMP} - \mathbf{w}^*_{BPM}) = \mathbf{e}_{AMP}F_{iB}G_{iP,M}. \tag{35}$$

Of course, local strain measures suit, instead, best ephemeral continua; they are, however, contingent on observer, so that one must resort to their orthogonal invariants. Some such have plain physical meaning: the six invariants of C and N, say, and

$$\alpha := \det X = \det \tilde{X}, \tag{36}$$

which measures the ratio of different changes of volume consequent or the affine deformations spelled out by F and G (or the inverse of the "protrusion" of motes in excess of that implied by F).

Actually, beyond α, a suitable, thorough referential measure of strain, is provided by the symmetric tensor

$$R'^T\tilde{X}^T\tilde{X}R' = N^{-\frac{1}{2}}CN^{-\frac{1}{2}}, \tag{37}$$

the determinant of which equals α^2, or, rather, by the traceless tensor of *distemper*

$$P := \frac{1}{\mathrm{tr}(CN^{-1})}N^{-\frac{1}{2}}CN^{-\frac{1}{2}} - \frac{1}{3}I, \tag{38}$$

which vanishes if and only if $C = \alpha^{\frac{2}{3}}N$. There is an analogy between this definition and one exploited within the theory of nematic liquid crystals [2]. From that theory one can import the two relevant scalar quantities: prolation and optical biaxiality, which can be expressed in terms of the eigenvalues of the tensor of distemper.

An inhomogeneity in a field of P, measured by its gradient, may effect a flux of P of intensity per unit volume gauged by the divergence of the gradient, so that the relevant quantity becomes the tensor

$$M := \Delta P = \mathrm{Div}(\mathrm{Grad}\, P) \in \mathrm{Sym}; \tag{39}$$

its second gradient $\mathbf{P}$ is also of interest:

$$\mathbf{P} := \mathrm{Grad}^2 M \in \mathrm{Sym} \otimes \mathrm{Sym}. \tag{40}$$

We observe that the distemper P is invariant for observer changes and that M and $\mathbf{P}$ offer adequate second- and fourth-order strain measures, respectively. Similar fields can be defined in local form by using strains (26).

3 Straining

Stretching and spin can be summoned for both macro- and meso-motions, for the first through L and, for the second, through B:

$$D = \mathrm{sym}\, L, \quad S = \mathrm{skw}\, L, \quad D^m = \mathrm{sym}\, B, \quad S^m = \mathrm{skw}\, B, \tag{41}$$

together with the *deranging* $(L - B)$ and its trace σ; we have that $\mathrm{sym}(\cdot) := \frac{1}{2}\left[(\cdot) + (\cdot)^T\right] \in \mathrm{Sym}$, the collection of second-order symmetric tensors, and $\mathrm{skw}(\cdot) := \frac{1}{2}\left[(\cdot) - t(\cdot)^T\right] \in \mathrm{Skw}$, the collection of second-order skew tensors.

D and D^m are not affected by a motion of the observer; a separate quantity with the same property is the skew part of the deranging, equal to the difference between vorticities

$$S^d = \mathrm{skw}\,(L - B) = S - S^m. \tag{42}$$

Whereas the symmetric part of $(L - B)$ measures the "slippage" of Eckart-Truesdell; the trace σ of the deranging measures suffusion. At a deeper level a third-order tensor $\mathbf{b}$ intrudes:

$$\mathbf{b} = \mathrm{grad}\, B. \tag{43}$$

Actually, mirroring the symmetry of $\mathbf{n}$ and $\tilde{\mathbf{n}} = \operatorname{grad}\tilde{N}$ with respect to their first two indices, it is rather the gradient of D^m which has a decisive rôle in the left-symmetric part $\mathbf{b}^s$:

$$\mathbf{b}^s = \frac{1}{2}(\mathbf{b} + {}^t\mathbf{b}) = \operatorname{grad} D^m, \tag{44}$$

here, alternatively to our use in equation (30) the exponent t on the left indicates transposition of the first two indices. A similar relation ties up the left-skew part $\mathbf{b}^a$ to the gradient of S^m:

$$\mathbf{b}^a = \frac{1}{2}(\mathbf{b} - {}^t\mathbf{b}) = \operatorname{grad} S^m. \tag{45}$$

Easy reasoning leads to the expression of strain rates from (22) and (24)

$$\dot{C} = 2F^T DF, \quad \dot{N} = 2G^T D^m G, \quad \dot{X} = G^{-1}(L-B)F, \tag{46}$$

$$\dot{Q} = R'^T(\dot{R}R^T - \dot{R}'R'^T)R = (R'^T\dot{R}')^T Q + Q(R^T\dot{R}) \tag{47}$$

and

$$\dot{Q}Q^T = R'^T(\dot{R}R^T - \dot{R}'R'^T)R'. \tag{48}$$

Notice also that

$$\operatorname{tr}(\dot{X}X^{-1}) = \operatorname{tr}(L-B) = \sigma = \frac{1}{2}\left(\dot{C}\cdot C^{-1} - \dot{N}\cdot N^{-1}\right). \tag{49}$$

Slightly more involved is the derivation of the simple relation between $\operatorname{grad} D^m$ and $\dot{\mathbf{n}}$

$$\dot{\mathbf{n}} = 2\operatorname{Grad}(G^T D^m G) = 2[\operatorname{grad}(G^T D^m G)]F. \tag{50}$$

Because, in the expression for $\dot{X}$, the symmetric part of $(L-B)$ can already be written in terms of $\dot{C}$ and $\dot{N}$ by $(46)_{1,2}$

$$\operatorname{sym}(L-B) = \frac{1}{2}F^{-T}(\dot{C} - X^T\dot{N}X)F^{-1}; \tag{51}$$

the really relevant term is the skew part of $(L-B)$, which can be obtained from equations $(23)_{1,2}$; in fact we have

$$L - B = \dot{F}F^{-1} - \dot{G}G^{-1} = \left(\dot{R}R^T - \dot{R}'R'^T\right) + \frac{1}{2}\left(RC^{-\frac{1}{2}}\dot{C}C^{-\frac{1}{2}}R^T - R'N^{-\frac{1}{2}}\dot{N}N^{-\frac{1}{2}}R'^T\right), \tag{52}$$

where the first tensor is skew and the second one symmetric. Thus, from relations (42), $(46)_2$ and (48), we read

$$S^d = \operatorname{skw}(G\dot{X}F^{-1}) = R'\dot{Q}R^T = \dot{R}R^T - \dot{R}'R'^T. \tag{53}$$

Thus a mesovorticity may be present, even if the macromotion were irrotational.

The connection between the time-derivatives $\dot{\mathbf{n}}$ and $\dot{W}$ of $\mathbf{n}$ and W, respectively, and the third-order tensor $\mathbf{b}$ is more remote. From (44), $(46)_{1,2}$ and (8), a slightly simpler expression obtains for the tensor $\mathbf{b}^s$

$$2\,\mathbf{b}^s_{ijk} = 2\,D^m_{ij,k} = G^{-1}_{Bi}\,\dot{\mathbf{n}}_{ABC}\,G^{-1}_{Aj}\,F^{-1}_{Ck} - \dot{N}_{AB}\,G^{-1}_{Bl}\,G_{lC,k}\left(G^{-1}_{Ai}\,G^{-1}_{Cj} + G^{-1}_{Aj}\,G^{-1}_{Ci}\right), \tag{54}$$

while, from (45), (9), $(23)_2$ and $(25)_2$, we have for the third order tensor $\mathbf{b}^a$

$$\mathbf{b}^a = \operatorname{grad}(\dot{R}'R'^T) = -\operatorname{grad}(\mathbf{e}\,\dot{q}') = -\mathbf{e}\left[\left(\operatorname{Grad} q'\right)^{\cdot} F^{-1}\right] = -\mathbf{e}\left(\dot{W}'F^{-1}\right). \tag{55}$$

We have already remarked that a rôle similar to that of L and B could be imputed to H, so that it is not outlandish to pull it back also onto the phantom reference. In fact, a tensor $H_* = G^{-1}HG^{-T}$ was already introduced elsewhere and the rate $\dot{H}_*$ plays, on occasion, a rôle similar to that of $\dot{C}$, $\dot{N}$, etc.

We record below expressions of the time derivatives of $\tilde{C}$, $\tilde{N}$ and $\tilde{X}$

$$(\tilde{C})^{\cdot} = 2\operatorname{sym}(L\tilde{C}), \quad (\tilde{N})^{\cdot} = 2\operatorname{sym}(B\tilde{N}), \quad (\tilde{X})^{\cdot} = L\tilde{X} - \tilde{X}B, \tag{56}$$

the latter being bound to σ by

$$\sigma = (\det \tilde{X})^{-1}\dot{\overline{(\det \tilde{X})}} = \frac{\dot{\alpha}}{\alpha} \tag{57}$$

by definition (36).

We should have mentioned also the time-derivatives of $\tilde{Q}$

$$(\tilde{Q})^{\cdot} = \dot{R}R^T\tilde{Q} + \tilde{Q}(\dot{R}'R'^T)^T \quad \text{leading to} \quad (\tilde{Q})^{\cdot}\tilde{Q}^T = \dot{R}R^T + \tilde{Q}(\dot{R}'R'^T)^T\tilde{Q}^T. \tag{58}$$

Truesdell's kinematic vorticity number υ can be read both for the macro and micromotion

$$\upsilon = \sqrt{\frac{S\cdot S}{D\cdot D}}, \quad \upsilon^m = \sqrt{\frac{S^m\cdot S^m}{D^m\cdot D^m}}, \tag{59}$$

and also for the deranging, when $D \neq D^m$,

$$\upsilon^d = \sqrt{\frac{(S-S^m)\cdot(S-S^m)}{(D-D^m)\cdot(D-D^m)}}. \tag{60}$$

When the latter vanishes, macro and microstraining differ by a stretching.

The kinetic fields quoted so far are insufficient to study fully the motions possible in ephemeral continua. Another essential field is Reynolds' tensor H; it measures the balanced cross-flow of motes or their balanced bouncing.

A terse rendition of H is based on its being symmetric and definite, so that it can be written in terms of eigenvalues χ_s^2, with $|\chi_s| = 1$, and unit eigenvectors h_s (s = 1, 2, 3)

$$H = \sum_{s=1}^{3} \chi_s^2\, h_s \otimes h_s. \tag{61}$$

Thus, of H could be given the following sketch: it can be regarded as generated by the balanced and unhindered cross-flow of six countervailing and equally populous clusters of molecules with velocities $((\sqrt{2})^{-1}\chi_s\, h_s)$ and $(-(\sqrt{2})^{-1}\chi_s\, h_s)$ (s = 1, 2, 3) with χ_s non-negative square root of χ_s^2. Such metaphor justifies the attribution to H the meaning of a measure of balanced cross-over rate. Alternatively one could imagine the same sums counting the number of bounces due to collisions.

4 Bundle streak lines

A sweeping question was often raised about the system (16) to (20): was the facing the inherent complexities really worthwhile in order to respond to (for some readers seemingly marginal) perplexities met in standard approaches?

We have added (in common with many earlier proposals) the affine mesomotion via the field B; but, finally, also the field of H. In fact it is the insertion of that last field that tells apart radically our scheme from others, an insertion intrinsic to the theory. So much so that motions allowed by the standard theory of continua would be granted here, when F coincides with G (and B with L; hence the vanishing of σ), only if artificially and incongruously H were imposed to vanish. The agitation of molecules within the loculus embodied by non-null c implies inextricably some (here only pseudo-thermal) effects.

To offer an almost baffling scene of H we have suggested above a spectral splitting of that symmetric positive definite tensor. It seemed appropriate to display a consequent alternative to the wind streaks. The latter are macroscopically obvious and even, within their limits, efficacious; however, at times, they seem to be less revealing, we think, than the 'bundle streaks' treated below.

For simplicity and to make those events more evident, we consider below only stationary flows where the only fields involved are those of v and H and balanced and unhindered cross-over of molecules is prominently influential. The reasoning is purely kinetic; the laws of interaction are simply assumed to provide the balances.

Then certain paths wholly separate from wind streaks (paths which we call 'bundle streaks') seem to describe incidents more vividly. In general, contrary to what happens for wind streaks, more than one such streak goes through each place within the body evidencing cross-flow.

The picture we intend to draw issues from imagining the set of molecules transiting across each place partitioned into six bundles each one characterized by its own peculiar velocity. The latter is defined via the filtered velocity v and the spectral version (61) of the tensor H:

$$\frac{1}{2}\left[\left(\sum_{i=1}^{3}\chi_i\right)^{-1}\chi_s\right]v \pm \left(\frac{1}{\sqrt{2}}\chi_s\right)h_s, \quad s = 1,2,3. \tag{62}$$

The six sets of paths are obtained by a process akin to that pursued in the definition of wind streaks, relying now on the six vector fields (62).

We refrain from pursuing here the consequences in general nor do we enter into details. We simply show three figures of plane flows with v in the plane, only one non-vanishing eigenvalue χ of H and the corresponding eigenvector h again in the plane with perfect bouncing only on the constraining planes.

The first figure portrays a channel plug flow within two parallel planes with v parallel to the planes and the only relevant eigenvector normal to them. The second a similar flow but over a single horizontal plane of molecules as though they were acted upon by gravity. The third a sort of avalanche sliding, as though under gravity, over an inclined plane. As already said we leave to the reader the task of filling necessary complements.

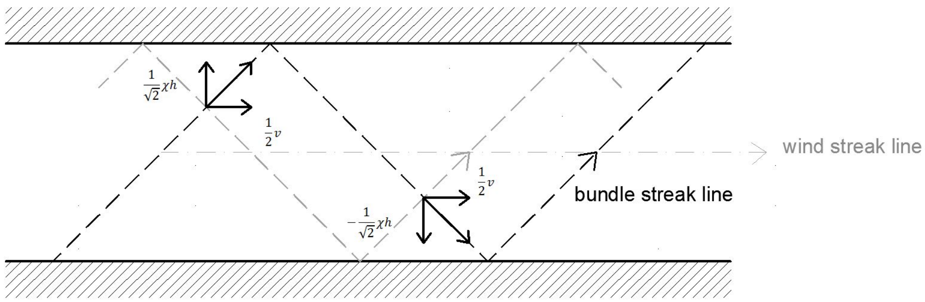

Fig. 1 Plug flow due to bouncing

The only goal was the obvious one of evidencing cases which would escape description without the agency of H; an agency which allows the study of phenomena were there is a continuum of shocks.

We restrict our account to the train of thought leading to the motion as envisioned on the first picture. Suppose that L and B be both null, but H be not null. Then v and G have the same value everywhere, though the motes flow against each other in a mass balanced mode; the gross motion is one of bare translation (say, a "plug flow" in a channel). Imagine the flow in a plane containing v. Possibly the flow of a fluid occupying the slab between two planes parallel to each other and to v; and observe the motion on a plane normal to the slab and, as said, containing v. Let h be the normal to the confining planes and restrict attention to the case when h is eigenvector of H and the corresponding constant eigenvalue χ^2 is the only non null one. Then the motion can be imagined as due to the bouncing of motes between the planes; at each place, two tribes of motes cross, equally populous, one with velocity $\left(\frac{1}{2}v + \frac{\chi h}{\sqrt{2}}\right)$, the other with velocity $\left(\frac{1}{2}v - \frac{\chi h}{\sqrt{2}}\right)$. At the point of contact only one bundle streak line transits and the shock offered by the confining planes causes a discontinuity of the otherwise penetrating addendum.

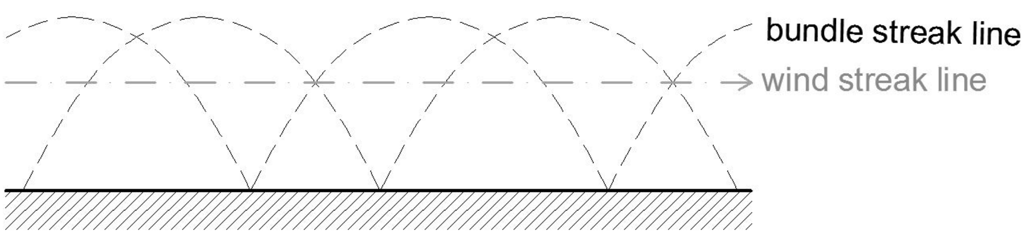

Fig. 2 Bouncing on a horizontal plane of heavy molecules

In Figure 2 the medium, assuming to be affected by gravity, is supported by a horizontal plane and lives in a vacuous environment devoid of molecules. A sort of near surface tension impedes the molecules to escape; such effect being mirrored thus: the only non-null eigenvalue decreases in the approach to the surface and becomes null there.

Figure 3 feigns to represent an avalanche with the heavy medium flowing along an inclined plane but otherwise behaving as under the pretended conditions described above.

A final due remark stems from recalling that, within our speculations, molecules are supposed to be all alike; therefore undisturbed balanced cross-over, imagined above, may be replaced by internal bouncing at encounters and, consequently, streaks

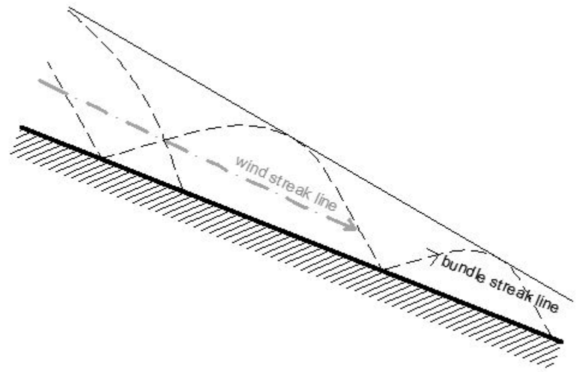

Fig. 3 Avalanche of heavy molecules on an inclined plane

(smooth except for the recoils at the confining border) may be replaced by streaks zigzagging within each loculus, the bouncing velocity being given by $(\chi h/\sqrt{2})$. At the macro level the shocks cover the continuum.

5 Energies

Primarily, we must muse upon a remark in §3 of [4] which evidences the fact that the kinetic energy tensor $\mathscr{W}$ of the cluster of molecules in each $\mathfrak{c}(x)$,

$$\mathscr{W} = \frac{1}{2\omega}\int_{\mathfrak{c}(x)} dy \int_{\mathscr{V}} \Theta(\tau,x;y,w)\, w\otimes w\, dw = \frac{1}{2\omega}\int_{\mathscr{V}} \hat{\theta}(\tau,x;y)\, w\otimes w\, dw, \quad \text{with} \quad \hat{\theta} = \int_{\mathfrak{c}(x)} \Theta\, dy, \tag{63}$$

cannot be determined in terms of filtered variables based on number densities alone: the distribution Θ has a direct rôle. Besides, the enormous body of results from the kinetic theory, stemming from the Boltzmann equation, proves, by implication, that Θ might be essential also to specify compulsion densities, be they volumetric or tactile.

Roughly, a parting of the ways occurs: either one restricts ambitions to what a sort of extended mechanics can offer by widening the range of descriptive parameters (as, here, where G and H are added) or one faces thermodynamic complexities, where the account of each local setting involves distribution functions directly not only through their filtered versions.

We take here the first way out and hence refer to (16)-(20); the consequent reduced kinetic energy tensor $\mathscr{W}_*$ per unit mass (already mentioned in [4]) is introduced as follows:

$$\mathscr{W}_* = \frac{1}{2\omega}\int_{\mathfrak{c}(x)} dy \int_{\mathscr{V}} \Theta(\tau,x;y,w)\, w_*\otimes w_*\, dw = \frac{1}{2\omega}\int_{\mathfrak{c}(x)} \theta(\tau,x;y)\, w_*\otimes w_*\, dy, \tag{64}$$

and found to be equal to

$$\mathscr{W}_* = \frac{1}{2}(v\otimes v + BYB^T + H) \tag{65}$$

whereas its Oldroyd derivative is equal to

$$\overset{\circ}{\overline{\mathscr{W}_*}} = \frac{1}{2}\dot{\overline{(v\otimes v)}} + \frac{1}{2}\overset{\circ}{\overline{(BYB^T + H)}} \tag{66}$$

(see comments on extensive and intensive variables after balance equations in Section 1).

By multiplying tensorially both members of (17) on the left by v and taking the symmetric component of the resulting tensors; acting with B on both sides of (19) and again taking the symmetric component; adding term by term the two results plus (20) multiplied on each side by $1/2$ and integrating the result over any fit region $\mathfrak{b}$ belonging to $\mathscr{B}_\tau$, the theorem of tensor kinetic energy follows:

$$\int_{\mathfrak{b}} \rho \overset{\circ}{\overline{\mathscr{W}_*}}\, dx = \int_{\mathfrak{b}} \rho \left[\mathrm{sym}(v\otimes b + BO) + \frac{1}{2}J\right] dx + \tag{67}$$

$$+\int_{\partial\mathfrak{b}}\left\{\mathrm{sym}[v\otimes Tn+B(\mathbf{m}n)]+\frac{1}{2}\mathbf{j}n\right\}d\mathscr{A}-\int_{\mathfrak{b}}\left[\mathrm{sym}(LT^T+BA+\mathbf{b}\mathbf{m}^t)+\frac{1}{2}Z\right]dx,$$

where $d\mathscr{A}$ is the surface measure over the boundary $\partial\mathfrak{b}$ of $\mathfrak{b}$ oriented by the unit normal n.

Before we proceed we must pause to reflect on how to express properly the roles of the tensors Z and $\mathbf{j}$. They must exhibit respectively power and power flux; the choice of such expressions is largely open and one may even raise doubts about the expediency of proposing one choice even of their basic constitutive aspects. We guess that they could be exhibited as products of some compulsion by the time-rate of change of some distortion and the choice of factors should reasonably precede constitutive proposals for them. What we say now, in other words, is that the constitutive decisions should be made in two stages. The abrupt astute choice for Z in [4] conforms nevertheless with the requirement above, but that for $\mathbf{j}$ it does not. So we restart anew.

As far as Z is concerned, it may be structured linearly as follows

$$Z=\mathbf{Z}E, \tag{68}$$

with the tensor E chosen, for instance, as L or B or U. Alternatively, and even better, directly as a time-derivative of C, N, X, U and M, with $\mathbf{Z}$ as a fourth-order tensor (symmetric in the first two indices), to be suggested itself constitutively, for instance, as a function of variables of strain mentioned in the section so named. Also H might, possibly, have been added as an entry. We did so in the second (44) of [7], leading, however to a troublesome non-linear dependence on rates. We avoid such dependence here, though the restriction is not at all compulsory, in general, but decisive for the developments of this section. So $\mathbf{Z}$ does not depend on strainings.

The third-order tensor $\mathbf{j}$ may be shaped similarly, though involving a higher order tensor (symmetric in the first two indices), say Γ again presumed (as we did for $\mathbf{Z}$) to depend on strain measures only, such a tensor acting on the gradient of E:

$$\mathbf{j}=\Gamma(\mathrm{grad}\,E). \tag{69}$$

Here resides the main earlier deficiency; as a consequence of the divergence theorem an extra term intrudes in the expression of the power of internal compulsions entailing the divergence of $\mathbf{j}$. Employing the choice (69) we have that

$$\begin{aligned}(\mathrm{div}\,\mathbf{j})_{ab} &= \mathbf{j}_{abc,c}=\left[\Gamma_{abcdef}E_{de,f}\right]_{,c}=\Gamma_{abcdef,c}E_{de,f}+\Gamma_{abcdef}E_{de,fc}=\\ &=\left(\Gamma_{abcdef,c}E_{de}\right)_{,f}-\Gamma_{abcdef,cf}E_{de}+\Gamma_{abcdef}E_{de,fc},\end{aligned} \tag{70}$$

and introducing the choices (68) and $(70)_5$ in the last two integrals of (67) and using again the divergence theorem we get

$$\begin{aligned}\int_{\mathfrak{b}}\rho\overset{\circ}{\overline{\mathscr{W}_*}}dx &= -\int_{\mathfrak{b}}\left\{\mathrm{sym}\left(LT^T+BA+\mathbf{b}\mathbf{m}^t\right)+\frac{1}{2}\left[\left(\mathbf{Z}+\mathrm{div}^2\hat{\Gamma}\right)E-\hat{\Gamma}\left(\mathrm{grad}^2E\right)\right]\right\}dx+\\ &+\int_{\mathfrak{b}}\rho\left[\mathrm{sym}(v\otimes b+BO)+\frac{1}{2}J\right]dx+\int_{\partial\mathfrak{b}}\left\{\mathrm{sym}[v\otimes Tn+B(\mathbf{m}n)]+\frac{1}{2}(\mathrm{div}\hat{\Gamma})(E\otimes n)\right\}d\mathscr{A},\end{aligned} \tag{71}$$

with

$$\hat{\Gamma}_{abdefc}:=\Gamma_{abcdef}. \tag{72}$$

The usual interpretation applies of the three terms of the right–hand side; external bodily compulsions, the first; external compulsions across the boundary, the second. Finally the last term is figured to measure the tensor power of internal compulsions; its density $\mathscr{P}^{in}$ is then

$$\mathscr{P}^{in}=-\left\{\mathrm{sym}\left(LT^T+BA+\mathbf{b}\mathbf{m}^t\right)+2^{-1}\left[\left(\mathbf{Z}+\mathrm{div}^2\hat{\Gamma}\right)E-\hat{\Gamma}\left(\mathrm{grad}^2E\right)\right]\right\}. \tag{73}$$

Now, our mathematical model of events must mirror rigorously the physical fact that, per se, any kinetic behavior of the frame, on which the observers sit, must have no influence whatsoever on their evaluation of the power density expressed either by its tensorial form (73) or, at least, by the trace of (73)

$$\mathrm{tr}\left(\mathscr{P}^{in}\right)=-\left\{T\cdot L+A^T\cdot B+{}^t\mathbf{m}\cdot\mathbf{b}+2^{-1}\left[\mathrm{str}\left(\mathbf{Z}+\mathrm{div}^2\hat{\Gamma}\right)\cdot E-\mathrm{str}\hat{\Gamma}\cdot\left(\mathrm{grad}^2E\right)\right]\right\}, \tag{74}$$

where the operator semitrace $\mathrm{str}(\cdot)$ contracts the first two indices of the operand, that is: $\mathrm{str}(\cdot)_{kl\ldots m}=(\cdot)_{iikl\ldots m}$.

Denoting by $\tilde{Z}$ the second-order tensor obtained from $\mathbf{Z}$ by taking half of its left semitrace and by $\mathbf{G}$ the fourth-order tensor (symmetric in the second set of two indices) obtained from $\hat{\Gamma}$ by taking half of its left semitrace, that is

$$\tilde{Z} = \frac{1}{2}\operatorname{str}\mathbf{Z} \quad \text{and} \quad \mathbf{G} = \frac{1}{2}\operatorname{str}\hat{\Gamma}, \tag{75}$$

then we have this alternative expression for the internal power trace (74):

$$\operatorname{tr}\left(\mathscr{P}^{in}\right) = -\left[T\cdot L + A^T\cdot B + {}^t\mathbf{m}\cdot\mathbf{b} + \left(\tilde{Z} + \operatorname{div}^2\mathbf{G}\right)\cdot E - \mathbf{G}\cdot\left(\operatorname{grad}^2 E\right)\right]. \tag{76}$$

But their reading of L and B are both changed by the addition of the same skew tensor as a consequence of a rigid rotation of their observer frame, a rotation which leaves, instead, $\mathbf{b}$ unscathed. Additional effects may ensue from certain choices of constitutive laws for T, A, $\mathbf{m}$, $\mathbf{Z}$ and Γ. Vice versa a cute constitutive rule might correct other unwanted influences. Thus we must delimit somehow the choices open to the mathematical modeller of constitutive laws for T, A, $\mathbf{Z}$ and Γ, specifically regarding the variables involved in those laws. In line with the results sought here we will assume that the variables be the strains mentioned in the section so named, thus avoiding the presence of time rates and consequent non-linearities, although we thus exclude effects of viscosity, for instance.

In any case, if the last two addenda are frame-indifferent together with the constitutive laws of T and A separately (as it is the case when the strain measure E is chosen independent from other strain measures in (74)), then the requirement of frame-indifference of (74) imposes the restriction

$$\operatorname{skw} T = \operatorname{skw} A. \tag{77}$$

The kinetic energy theorem (67) could be imagined as due to the conjoining of two theorems, one merely issuing from (17) and formally involving only standard quantities

$$\begin{aligned}\frac{1}{2}\int_{\mathfrak{b}}\rho\left[(v\otimes v)^{\cdot} + \sigma\, v\otimes v\right]dx = -\int_{\mathfrak{b}}\operatorname{sym}(LT^T)\,dx + \\ + \int_{\mathfrak{b}}\rho\operatorname{sym}(v\otimes b)\,dx + \int_{\partial\mathfrak{b}}\operatorname{sym}(v\otimes Tn)\,d\mathscr{A}\end{aligned} \tag{78}$$

and the other counting contributions due to the new fields entering our extended mechanics

$$\begin{aligned}\frac{1}{2}\int_{\mathfrak{b}}\rho\,\overline{(BY\overset{\circ}{B^T} + H)}\,dx = -\int_{\mathfrak{b}}\left\{\operatorname{sym}(BA + \mathbf{b}\mathbf{m}^t) + \frac{1}{2}\left[\left(\mathbf{Z} + \operatorname{div}^2\hat{\Gamma}\right)E - \hat{\Gamma}\left(\operatorname{grad}^2 E\right)\right]\right\}dx + \\ + \int_{\mathfrak{b}}\rho\left[\operatorname{sym}(BO) + \frac{1}{2}J\right]dx + \int_{\partial\mathfrak{b}}\left\{\operatorname{sym}[B(\mathbf{m}n)] + \frac{1}{2}(\operatorname{div}\hat{\Gamma})(E\otimes n)\right\}d\mathscr{A},\end{aligned} \tag{79}$$

both valid separately, though with strong links between them.

Under becoming assumptions of smoothness and in view of its validity for any choice of $\mathfrak{b}$, (79) may be replaced by a prerequisite on densities

$$\begin{aligned}\rho\,\overline{(BY\overset{\circ}{B^T} + H)} + 2\operatorname{sym}(BA + \mathbf{b}\mathbf{m}^t) + \left(\mathbf{Z} + \operatorname{div}^2\hat{\Gamma}\right)E - \hat{\Gamma}\left(\operatorname{grad}^2 E\right) = \\ = \rho\left[2\operatorname{sym}(BO) + J\right] + \operatorname{div}\left[B\oslash\mathbf{m} + {}^t(B\oslash\mathbf{m}) + (\operatorname{div}\hat{\Gamma})^{2t}E\right],\end{aligned} \tag{80}$$

where the product $\oslash$ between tensors of the second and the third order is so defined: $(B\oslash\mathbf{m})_{ijl} := B_{ik}\mathbf{m}_{kjl}$, while the double right transposition $(\cdot)^{2t}$ means: $\Omega^{2t}_{abfde} = \Omega_{abdef}$.

Then we could interpret (80) as the balance equation for the whole meso energy: *i.e.*, comprising also the affine (in particular the vorticose) contribution beyond the sole energy of agitation H. In fact, nominally, (80) could substitute (20) within the set of balance equations.

Such perspective would become particularly suggestive if the medium were perfect in the sense that a potential energy exists from which all compulsions can be derived so that the whole left–hand side of (80) may be written as the time–rate of an internal meso energy. The matter is discussed in some detail in [4], see especially the final part of §8. There are already precise hints in the literature regarding A and $\mathbf{m}$; a suggestion for Z advanced in §8 of [4] was already quoted; clearly the presence of viscous effects as specified by $(44)_2$ of [7] must instead be excluded. Finally (80) would mimic, on the face of it, at the meso level, the first principle of thermodynamics, and a class of ‘thermal’ phenomena could be dealt with within our extended mechanics.

6 Special classes of ephemeral continua

There are classes of ephemeral continua which either mimic some classes already known in the literature or show unexpected details; alternatively their study makes progress in following sections easier. We collect a few prompted relevant remarks here.

6.1 Affine mesostructure

A demand was broached for a check that the well-known theory of continua with affine microstructure (see, for instance, [9]) may actually be included in ours, perhaps under specified appropriate assumptions. Now, the balance equations for affine bodies do not involve the tensor H at all. Equation (20) must be identically satisfied by a vanishing H; hence $J, Z, \mathbf{j}$ must be null. Such special choice is not precluded in our theory, though only some corollaries implying non vanishing H may loose validity.

A delicate matter, instead, is the decision regarding the relation between our tensor G and the tensor, say G^*, of the affine deformation. One must declare a physical interpretation attributed to quantities involved. Firstly, does G^* exist at all or does the theory address itself to higher order phenomena, as in a theory of dislocations, not admitting that existence? Even if G^* exist, it could have been introduced to describe events (non-mechanical, for instance) without any direct bearing on the gross motion. Under both circumstances we are trying to compare differently addressed theories and the challenge does not make sense.

Finally, does G^* determine a change of shape of the material element beyond that consequent to the basic motion (spelled out by our tensor F and deriving from the filtered velocity v)? Then it is the theory of affine bodies which needs revision as it is easy to show. In fact our G becomes equal to G^*F and $\dot{G}G^{-1} = \dot{G}^*(G^*)^{-1} + G^*L(G^*)^{-1}$ and, taking the traces, $\sigma = \mathrm{tr}(L-B) = \mathrm{tr}[\dot{G}^*(G^)*^{-1}]$, so that rate of change of volume must enter the theory of affine bodies. Of course, if G^* reduces to an orthogonal tensor, as is often the considered case, then $\ddot{G}G^{-1}$ is skew and its trace vanishes.

More delicate would be the discussion for the multipolar continuum of Green & Rivlin [16], as, for them, the mesostructure is more complex.

6.2 Wind streaks mesostructure

In the same vein of the preceding paragraph we thought of importing some wisely chosen constraints reducing the number of fields ultimately relevant and of exploring if they might be perfect, i.e. calling only for powerless reactions. The most conspicuous example is one requiring wind streak lines to coincide with particle paths: $G = F$ or $B = L$, for which the suffusion σ is always null. Although seeming to flout the key grounds for introducing the full theory, the constraint preserves, nevertheless, one distinguishing feature: the dominant presence of the tensor H and the corresponding compulsions.

Essential repercussions attend the choices for Z and $\mathbf{j}$, together with some simplifying constraints . We persevere in the options (68), (69) and the consequent expression (76) of power. But we impose a crucial restriction: the compulsions $\mathbf{Z}$ and Γ depend at most on the strains C, N, Q, W, M, $\mathbf{n}$, $\mathbf{P}$ and NOT on the associated strainings (including H); the weight of this stricture will transpire obliquely when discussing hypoelastic bodies.

There remains the choice of E for which we propose one very direct, D for E, therefore, for constitutive laws (68) and (69), $\mathbf{Z}$ must be symmetric with respect to last two indices and Γ with respect to the fourth and the fifth index, and so, for definitions (72) and (75), we have

$$\tilde{Z} \in \mathrm{Sym} \quad \text{and} \quad \mathbf{G} \in \mathrm{Sym} \otimes \mathrm{Lin}, \tag{81}$$

where we denote with Lin the collection of second-order tensors.

To deduce the elementary implications of a general, frame-indifferent, linear constraint on L and B, we follow the derivations of the theory of constrained ephemeral continua provided by Capriz, Fried and Seguin [6], as used, for example, to NavierStokes-$\alpha\beta$ continua provided in [5]. This approach rests on considerations involving the power of the internal actions (76). Following the traditional approach to dealing with constraints, we suppose that all compulsions are supposed to be sums of an active term specified by a constitutive rule and a reactive term. The set of the latter being collectively such to lead in toto to a vanishing power for all strainings permitted by the constraint. Symbols used for the separate addenda will be the usual ones affected by the subscript a or r.

Then the internal power is given by (76), changing E into D and using (77),

$$\mathrm{tr}\left(\mathscr{P}^{in}\right) = -\left[\mathrm{sym}\left(T + A^T + \tilde{Z} + \mathrm{div}^2\,\mathbf{G}\right) \cdot D + {}^t\mathbf{m} \cdot \mathrm{grad}\,L - \mathbf{G} \cdot \left(\mathrm{grad}^2 D\right)\right]. \tag{82}$$

We call sdev $\hat{\Gamma}$ the deviatoric part of $\hat{\Gamma}$ based on str $\hat{\Gamma}$, and sdev $\mathbf{Z}$ similarly based on str $\mathbf{Z}$. As they do not enter this power, we presume that they do no react to the constraint and so they are both active, while each compulsion being now the sum of two as just mentioned:

$$T = T_a + T_r,\ A = A_a + A_r,\ \mathbf{m} = \mathbf{m}_a + \mathbf{m}_r,\ \tilde{Z} = \tilde{Z}_a + \tilde{Z}_r,\ \mathbf{G} = \mathbf{G}_a + \mathbf{G}_r,\ \mathrm{sdev}\,\hat{\Gamma}_r = 0,\ \mathrm{sdev}\,\mathbf{Z}_r = \mathbf{0}; \tag{83}$$

thus, for (81), we observe that

$$\mathrm{skw}\,\tilde{Z}_r = -\mathrm{skw}\,\tilde{Z}_a \quad \text{and} \quad \mathbf{G}_r^a = -\mathbf{G}_a^a, \tag{84}$$

where $(\cdot)^a$ is the left-skew part of $(\cdot)$ defined by (45).

The sum of quantities relating to reactions must vanish

$$\mathrm{sym}\left(T_r + A_r^T + \tilde{Z}_r + \mathrm{div}^2\,\mathbf{G}_r\right) \cdot D + {}^t\mathbf{m}_r \cdot \mathrm{grad}^2\, v - \mathbf{G}_r \cdot \mathrm{grad}^2\, D = 0, \quad \forall D \in \mathrm{Sym}, \quad \forall v \in \mathscr{V}; \tag{85}$$

therefore we must have

$$\mathrm{sym}\left(T_r + A_r^T + \tilde{Z}_r + \mathrm{div}^2\,\mathbf{G}_r\right) = 0, \quad {}^t\mathbf{m}_r \in \mathscr{V} \otimes \mathrm{Skw} \quad \text{and} \quad \mathbf{G}_r \in \mathrm{Skw} \otimes \mathrm{Skw}, \tag{86}$$

where we denote with Skw the collection of second-order skew tensors.

The final step is to seek the appropriate linear combination of equations 16)-(20) so that the issuing result avoids the presence of reactive terms. We add term by term to equation (17) the divergences of (19) and of (20) coming to a result where T_r, A_r and $\mathbf{m}_r$ do not appear

$$\rho\left[\frac{\partial v}{\partial \tau} + (\mathrm{grad}\, v)\, v\right] + \mathrm{div}\left\{\rho\left[H - \left(\frac{\partial L}{\partial \tau} + (\mathrm{grad}\, L)\, v\right) Y\right]\right\} = \rho b + \mathrm{div}\left[\mathrm{sym}\,(T_a + A_a) - \rho O^T - \mathrm{div}\,({}^t\mathbf{m}_a) - \mathrm{sym}\,\tilde{Z}_r\right], \tag{87}$$

because $\mathrm{div}^2\,({}^t\mathbf{m}_r) = \mathbf{0}$ and $\mathrm{div}^3\,\mathbf{G}_r = \mathbf{0}$ for relations $(86)_{2,3}$, respectively.

Instead, for relations $(83)_{6,7}$ and $(86)_3$, the last balance equation (20) reduces to:

$$\rho\left[\frac{\partial H}{\partial \tau} + (\mathrm{grad}\, H)\, v\right] = \rho J - \mathbf{Z}_a D + \mathrm{div}\,[\Gamma_a(\mathrm{grad} D)] - \frac{2}{3}\,(D \cdot \mathrm{sym}\,\tilde{Z}_r) I. \tag{88}$$

Regrettably it does not seem possible, by such elementary stratagem at least, to avoid the presence of the symmetric part of $\tilde{Z}_r$. Such stumbling-block is met even under simpler circumstances, such as when incompressibility is imposed on a Navier-Stokes fluid (see §4.6.2 of [18]). A frequent presumption, when applied to our case, is that forcing the molecules to obey a condition consequent the artificial invention of wind kinetics cannot be pursued without recourse to a coarse mechanism. The effect of the latter must be described by use of an additional constitutive rule: the reaction to the constraint becomes an addition to the active compulsion. Consequently reactive unknown addenda must vanish, but in (87) $\mathrm{sym}\,\tilde{Z}_r$ must be substituted by the active reaction $\mathrm{sym}\,\tilde{Z}_c$ of the constraint.

To summarize we have come to the balance equations

$$\frac{\partial \rho}{\partial \tau} + \mathrm{div}\,(\rho v) = 0, \tag{89}$$

$$\rho\left[\frac{\partial v}{\partial \tau} + (\mathrm{grad}\, v)\, v\right] + \mathrm{div}\left\{\rho\left[H - \left(\frac{\partial L}{\partial \tau} + (\mathrm{grad}\, L)\, v\right) Y\right]\right\} = \rho b + \mathrm{div}\left[\mathrm{sym}\,(T_a + A_a) - \rho O^T - \mathrm{div}\,({}^t\mathbf{m}_a) - \mathrm{sym}\,\tilde{Z}_c\right], \tag{90}$$

$$\rho\left[\frac{\partial H}{\partial \tau} + (\mathrm{grad}\, H)\, v\right] = \rho J - \mathbf{Z}_a D + \mathrm{div}\,[\Gamma_a\,(\mathrm{grad} D)] - \frac{2}{3}\,(D \cdot \mathrm{sym}\,\tilde{Z}_c) I. \tag{91}$$

We examine now the consequences of a second specific choice for E as the time derivative of the laplacian M of the distemper given by (39) in §2,:

$$Z = \mathbf{Z}\dot{M} \quad \text{and} \quad \mathbf{j} = \Gamma(\mathrm{grad}\,\dot{M}), \tag{92}$$

with $\mathbf{Z}$ symmetric also in the second two indices, as it must be Γ in the fifth and sixth one; then we have the following expression for the trace of the density of internal power (76):

$$\mathrm{tr}\left(\mathscr{P}^{in}\right) = -\left[T \cdot L + A^T \cdot B + {}^t\mathbf{m} \cdot \mathbf{b} + \left(\tilde{Z} + \mathrm{div}^2\,\mathbf{G}\right) \cdot \dot{M} - \mathbf{G} \cdot \left(\mathrm{grad}^2\,\dot{M}\right)\right]. \tag{93}$$

By imposing the perfect constraint $G=F$ we have that $N=C$ and so, from definition (38), $P=0$ for which $M=\Delta P=0$ and $\dot{M}=0$; therefore compulsions Z and $\mathbf{j}$ vanish for relations (92).

Also in this case the power relating to reactions is identically null for any motion allowed by the constraint:

$$(T_r + A_r^T)\cdot L + {}^t\mathbf{m}_r \cdot \mathrm{grad}^2 v = 0, \quad \forall L \in \mathrm{Lin}, \quad \forall v \in \mathscr{V}; \tag{94}$$

therefore we must have

$$T_r + A_r^T = 0 \quad \text{and} \quad \mathbf{m}_r + \mathbf{m}_r^T = \mathbf{0}. \tag{95}$$

The transpose of the balance of mesomomentum (19) determines $T_r = -A_r^T$ in the form

$$T_r = \rho\left\{\left[\frac{\partial L}{\partial \tau} + (\mathrm{grad}\, L)v\right]Y - H - O^T\right\} + A_a^T - [\mathrm{div}\,(\mathbf{m}_a + \mathbf{m}_r)]^T; \tag{96}$$

moreover, relations (95) assures us that

$$\mathrm{skw}\, T_r = \mathrm{skw} A_r \quad \text{and} \quad \mathrm{div}[(\mathrm{div}\mathbf{m}_r)^T] = \mathbf{0}, \tag{97}$$

and so, from (77), we have

$$\mathrm{skw}\, T_a = \mathrm{skw} A_a. \tag{98}$$

The resulting system of balance equations is now:

$$\frac{\partial \rho}{\partial \tau} + \mathrm{div}\,(\rho v) = 0, \tag{99}$$

$$\rho\left(\frac{\partial v}{\partial \tau} + Lv\right) + \mathrm{div}\left\{\rho\left[H - \left(\frac{\partial L}{\partial \tau} + (\mathrm{grad}L)v\right)Y\right]\right\} = \rho b - \mathrm{div}\left(\rho O^T\right) + \mathrm{div}\left[\mathrm{sym}(T_a + A_a) - (\mathrm{div}\,\mathbf{m}_a)^T\right], \tag{100}$$

$$\rho\left[\frac{\partial H}{\partial \tau} + (\mathrm{grad}H)\, v\right] = \rho J. \tag{101}$$

6.3 Liquids

Fluids and, in particular, liquids are identified via classes of constitutive laws and categories of essential constitutive variables (commonly L) with consequent prevalent but not strictly exclusive kinetic phenomena. Liquids, though, are usually supposed to obey besides to the constraint of incompressibility.

For extreme simplicity we restrict our investigation to the case where the hypotheses in the previous sub-section still hold (for a first analysis where such restriction $B=L$ is avoided see [3]). However, the final results fail to apply as a consequence of the extra constraint $\mathrm{div}\, v = \mathrm{tr}\, D = 0$, which requires D and L to be changed into dev D and dev L in (82), respectively.

In this case, the vanishing of the reactive internal power (85) brings us to

$$\mathrm{dev}\left(T_r + A_r^T + \tilde{Z}_r + \mathrm{div}^2\,\mathbf{G}_r\right)\cdot \mathrm{dev}\, D + {}^t\mathbf{m}_r \cdot \mathrm{grad}(\mathrm{dev} L) - \mathbf{G}_r \cdot \mathrm{grad}^2\,(\mathrm{dev} D) = 0, \quad \forall D \in \mathrm{Sym}, \quad \forall L \in \mathrm{Lin}, \tag{102}$$

that is

$$\mathrm{dev}\left(T_r + A_r^T + \tilde{Z}_r + \mathrm{div}^2\,\mathbf{G}_r\right) = 0, \quad \mathbf{m}_r \in \mathrm{Sph} \otimes \mathscr{V} \quad \text{and} \quad \mathbf{G}_r \in (\mathrm{Sph} \oplus \mathrm{Skw}) \otimes \mathrm{Skw}, \tag{103}$$

where Sph is the space of spherical tensors.

Now, by following the same developments to obtain (87), we have for relations (103) that the Cauchy equation for liquids is

$$\begin{aligned}\rho\left[\frac{\partial v}{\partial \tau} + (\mathrm{grad}\; v)\, v\right] + \mathrm{div}\left\{\rho\left\{\mathrm{dev}\left[H - \left(\frac{\partial L}{\partial \tau} + (\mathrm{grad}\; L)\, v\right)Y\right] - \mathrm{skw}\left(\frac{\partial L}{\partial \tau} + (\mathrm{grad}\; L)\, v\right)Y\right\}\right\} = \\ = \rho b + \mathrm{div}\left[\mathrm{sym}\, T_a + \mathrm{dev}\left(A_a - \mathrm{div}\,\mathbf{m}_a - \rho\, O^T\right) + \mathrm{skw}\left(\mathrm{div}\,\mathbf{m}_a - \rho\, O^T\right) + \mathrm{sph}\, T_r - \mathrm{dev}\,\tilde{Z}_r\right];\end{aligned} \tag{104}$$

while, for relations $(83)_{6,7}$ and $(103)_3$, the balance equation (20) for H reduces to:

$$\rho\left[\frac{\partial H}{\partial \tau} + (\mathrm{grad}\; H)\, v\right] = \rho J - \mathbf{Z}_a\, \mathrm{dev}\, D + \mathrm{div}\{\Gamma_a[\mathrm{grad}(\mathrm{dev} D)]\} - \frac{2}{3}\,(\mathrm{dev}\, D \cdot \mathrm{dev}\,\tilde{Z}_r)\, I. \tag{105}$$

The prevailing irrelevance of affine effects in liquids leads to the acceptance, for them, of the vanishing of Y, O, A_a and $\mathbf{m}_a$ leading to

$$\rho\left[\frac{\partial v}{\partial \tau}+(\operatorname{grad} v)\, v\right]+\operatorname{div}(\rho \operatorname{dev} H)=\rho b+\operatorname{div}\left[\operatorname{sym} T_a+T_r^*\right], \tag{106}$$

with $T_r^* := \frac{1}{3}(\operatorname{tr} T_r)I - \operatorname{dev}\tilde{Z}_r$.

The reaction stress T_r^* of the constraint must be described again by an additional constitutive rule. If the variables already involved be sufficient, together with H, to effect that rule or some perquisite accessory variables be needed is a matter we leave open for the moment. Again, the usual disregard of direct molecular effects leads to the acceptance of the vanishing of both H and $\tilde{Z}_r$, with the conclusion that the constraint of incompressibility introduces only an extra pressure equal to $-\frac{1}{3}\operatorname{tr} T_r$. As for the latter we may have to take a think from the stress $-(\lambda+\frac{2}{3}\mu)D$, present for compressible fluids, presuming that its trace has a finite limit $-(\lambda+\frac{2}{3}\mu)\tau_*^{-1}I$, when $\operatorname{tr} D \to 0$, where the parameter τ_* as time as physical dimension.

6.4 The perfect gas

The perfect gas could be classed as an energy preserving ephemeral continuum with wind-streak mesostructure for which the affine effects are missing (Y, A, O, $\mathbf{m}$ all vanish). Besides, tensors such as those involving $\mathbf{G}_a$ might have meaning only if other forms of energy were involved (as in the transitions to liquids).

Therefore, we use these hypotheses in the balance equations (89-(91) to obtain:

$$\frac{\partial \rho}{\partial \tau}+\operatorname{div}(\rho v)=0, \tag{107}$$

$$\rho\left[\frac{\partial v}{\partial \tau}+(\operatorname{grad} v)\, v\right]+\operatorname{div}(\rho H)=\rho b+\operatorname{div}\left[\operatorname{sym}(T_a-\tilde{Z}_c)\right], \tag{108}$$

$$\rho\left[\frac{\partial H}{\partial \tau}+(\operatorname{grad} H) v\right]=\rho J-\frac{2}{3}\left[\operatorname{sym}(\tilde{Z}_a+\tilde{Z}_c)\cdot D\right] I. \tag{109}$$

Then, in this context, the related kinetic energy theorem (71) would take the form

$$\begin{aligned}\int_{\mathfrak{b}} \rho \dot{\overline{\mathscr{W}_*}}\, dx = & -\int_{\mathfrak{b}}\left\{\operatorname{sym}\left\{L\left[\operatorname{sym}(T_a-\tilde{Z}_c)\right]\right\}+\frac{1}{3}\left\{D\cdot\left[\operatorname{sym}(\tilde{Z}_a+\tilde{Z}_c)\right]\right\} I\right\} dx+ \\ & +\int_{\mathfrak{b}} \rho\left[\operatorname{sym}(v\otimes b)+\frac{1}{2} J\right] dx+\int_{\partial\mathfrak{b}} \operatorname{sym}\left\{v\otimes\left[\operatorname{sym}(T_a-\tilde{Z}_c) n\right]\right\} d\mathscr{A},\end{aligned} \tag{110}$$

so that the internal power density (76) reduces to

$$\operatorname{tr}\left(\mathscr{P}^{in}\right)=-\operatorname{sym}\left(T_a+\tilde{Z}_a\right)\cdot D. \tag{111}$$

If it is also assumed that exists a potential ϕ, function only of C, then

$$\operatorname{sym}\tilde{Z}_a=\rho\frac{d\phi}{dC}-\operatorname{sym} T_a, \tag{112}$$

and so the balance (109) and the theorem (110) reduces to

$$\rho\left[\frac{\partial H}{\partial \tau}+(\operatorname{grad} H) v\right]=\rho J+\frac{2}{3}\left[\operatorname{sym}(T_a-\tilde{Z}_c)\cdot D-\rho\frac{d\phi}{dC}\right] I \qquad \text{and} \tag{113}$$

$$\begin{aligned}\int_{\mathfrak{b}} \rho \dot{\overline{\mathscr{W}_*}}\, dx = & -\int_{\mathfrak{b}}\left\{\operatorname{sym}\left\{L\left[\operatorname{sym}(T_a-\tilde{Z}_c)\right]\right\}+\frac{1}{3}\left\{\rho\frac{d\phi}{dC}-\left[\operatorname{sym}(T_a-\tilde{Z}_c)\right]\cdot D\right\} I\right\} dx+ \\ & +\int_{\mathfrak{b}} \rho\left[\operatorname{sym}(v\otimes b)+\frac{1}{2} J\right] dx+\int_{\partial\mathfrak{b}} \operatorname{sym}\left\{v\otimes\left[\operatorname{sym}(T_a-\tilde{Z}_c) n\right]\right\} d\mathscr{A}.\end{aligned} \tag{114}$$

Thus, in the system of balance equations, only the terms $\tilde{Z}_c$ require interpretation and a proposal, perhaps using thermal concepts.

7 Hyperelastic ephemeral continua

Again within this section we tailor to our present needs results partly published in the second part of §8 of [4], preface them with qualifying reflections which may change certain conclusions radically or, at least, open the way to alternatives, some of which, not mentioned earlier, we choose as the most appropriate to suggest specific, perhaps even more rewarding, developments. Essential repercussions attend the choices for Z and $\mathbf{j}$, together with some simplifying constraints as in Section 6.2. We persevere in the options (68), (69) and the consequent expression (76) of power. Notice, in the latter, that only $\tilde{Z}$ and G enter rather than the full $\mathbf{Z}$ and Γ. Thus, essentially to simplify processes, we will pursue henceforth the restricted goals achievable when (68) and (69) reduce to

$$Z = \frac{2}{3}(\tilde{Z}\cdot E)I \quad \text{and} \quad \mathbf{j} = \frac{1}{3}I\otimes[(\mathrm{str}\Gamma)\mathrm{grad}E], \tag{115}$$

so that, for equation (70),

$$\mathrm{div}\,\mathbf{j} = \frac{2}{3}\{\mathrm{div}[({}^t\mathrm{div}\,\mathbf{G})^T E] - (\mathrm{div}^2\mathbf{G})\cdot E + \mathbf{G}\cdot(\mathrm{grad}^2 E)\}I. \tag{116}$$

As we made in §6.2, we impose again the restriction that the compulsions $\mathbf{Z}$ and Γ depend at most on the strains C, N, Q, W, M, $\mathbf{n}$, $\mathbf{P}$ and NOT on the associated strainings (including H); to evidence two distinct salient classes of bodies we choose now L for E and successively the time derivative of the laplacian M, exploring some corollaries within these peculiar categories.

With the first choice the internal power (76) reduces to

$$\mathrm{tr}\left(\mathscr{P}^{in}\right) = -\left[\left(T+\tilde{Z}+\mathrm{div}^2\mathbf{G}\right)\cdot L + A^T\cdot B + {}^t\mathbf{m}\cdot\mathbf{b} - \mathbf{G}\cdot\left(\mathrm{grad}^2 L\right)\right]. \tag{117}$$

Independence of that power from the observer requires

$$\mathrm{skw}\,\left(T+\tilde{Z}+\mathrm{div}^2\mathbf{G}\right) = \mathrm{skw}\,A. \tag{118}$$

Now we follow usual developments to deal with the perfect constraint $B = L$ by using splittings $(83)_{1-5}$ and the vanishing of the reactive internal power, so that

$$\left(T_r + A_r^T + \tilde{Z}_r + \mathrm{div}^2\mathbf{G}_r\right)\cdot L + {}^t\mathbf{m}_r\cdot\mathrm{grad}^2 v - \mathbf{G}_r\cdot\mathrm{grad}^3 v = 0, \quad \forall L\in\mathrm{Lin}, \quad \forall v\in\mathscr{V}; \tag{119}$$

therefore we must have

$$T_r = -A_r^T - (\tilde{Z}_r + \mathrm{div}^2\mathbf{G}_r), \quad \mathbf{m}_r + \mathbf{m}_r^T = \mathbf{0}, \quad \text{and} \quad (\mathbf{G}_r)_{ijkl} + (\mathbf{G}_r)_{ikjl} + (\mathbf{G}_r)_{ijlk} = 0. \tag{120}$$

Relations $(120)_{2,3}$ assures us that, again, $\mathrm{div}^2({}^t\mathbf{m}_r) = \mathbf{0}$ and $\mathrm{div}^3\mathbf{G}_r = \mathbf{0}$, thus the Cauchy's balance equation (17) in the constrained case reduces to

$$\rho\left[\frac{\partial v}{\partial\tau} + (\mathrm{grad}\,v)\,v\right] + \mathrm{div}\left\{\rho\left[H - \left(\frac{\partial L}{\partial\tau} + (\mathrm{grad}\,L)\,v\right)Y\right]\right\} = \rho f + \mathrm{div}\left[T_a + A_a^T - \mathrm{div}({}^t\mathbf{m}_a) - \tilde{Z}_r\right], \tag{121}$$

where T_r, A_r and $\mathbf{m}_r$ do not appear and the effective specific external body force f is defined by

$$f = b - \mathrm{div}\left(O^T\right) - O^T\mathrm{grad}(\ln\rho). \tag{122}$$

The balance of moment of momentum (19) is now irrelevant to the theory, aside from determining the reaction T_r via $(120)_1$; moreover, this relation specializes (118) to

$$\mathrm{skw}\,\left(T_a + \tilde{Z}_a + \mathrm{div}^2\mathbf{G}_a\right) = \mathrm{skw}\,A_a. \tag{123}$$

As already remarked after equation (88), no obligation restricts the choice of $\tilde{Z}_r$ and $\mathbf{G}_r$, other than $(120)_3$ and one could propose constitutive laws for them, therefore the balance equations are

$$\frac{\partial \rho}{\partial \tau} + \mathrm{div}\,(\rho v) = 0\,; \tag{124}$$

$$\rho \left[\frac{\partial v}{\partial \tau} + (\mathrm{grad}\; v)\, v \right] + \mathrm{div}\, \left\{ \rho \left[H - \left(\frac{\partial L}{\partial \tau} + (\mathrm{grad}\, L)\, v \right) Y \right] \right\} = \rho f + \mathrm{div}\, \left[T_a + A_a^T - \mathrm{div}\,({}^t\mathbf{m}_a) - \tilde{Z}_c \right]; \tag{125}$$

$$\rho \left[\frac{\partial H}{\partial \tau} + (\mathrm{grad}\, H) v \right] = \rho J + \frac{2}{3} \left\{ \mathbf{G}_a \cdot \mathrm{grad}^2 L + [\mathrm{div}\,(\mathbf{G}_a + \mathbf{G}_c)] \cdot \mathrm{grad}\, L - (\tilde{Z}_a + \tilde{Z}_c) \cdot L \right\} I. \tag{126}$$

In this first choice the opposite of the internal power (117) with relations (120) and (123) reduces to

$$\begin{aligned}
-\mathrm{tr}\left(\mathscr{P}^{in}\right) &= \left\{ \left[\mathrm{sym}\left(T_a + A_a^T + \tilde{Z}_a\right) + \mathrm{div}^2\, \mathbf{G}_a^s \right] \cdot D + \mathbf{m}_a^s \cdot \mathrm{grad}\, D - \mathbf{m}_a^a \cdot \mathrm{grad}\, S - \mathbf{G}_a^s \cdot \mathrm{grad}^2 D - \mathbf{G}_a^a \cdot \mathrm{grad}^2 S \right\} = \\
&= \frac{1}{2} \left\{ F_{Ai}^{-1} F_{Bj}^{-1} \left[\mathrm{sym}\, \left(T_a + A_a^T + \tilde{Z}_a\right)_{ij} + \mathbf{G}^s_{ijkm,km} - F_{jC,k} \mathbf{m}^s_{ilk} F_{Cl}^{-1} \right] + 2\, \mathbf{G}^s_{ijkm} \left(F_{Ai}^{-1} F_{Bl}^{-1} F_{lC,k} F_{Cj}^{-1} \right)_{,m} \right\} \dot{C}_{AB} + \\
&+ \frac{1}{2} \left\{ F_{Ai}^{-1} F_{Bj}^{-1}\, \mathbf{m}^s_{ijk} F_{Ck}^{-1} - \mathbf{G}^s_{ijkm} \left[\left(F_{Ai}^{-1} F_{Bj}^{-1} F_{Ck}^{-1} \right)_{,m} - 2\, F_{Ai}^{-1} F_{Bl}^{-1} F_{lD,k} F_{Dj}^{-1} F_{Cm}^{-1} \right] \right\} \dot{\mathbf{c}}_{ABC} - \\
&- \frac{1}{2} \left(\mathbf{G}^s_{ijkm} F_{Ai}^{-1} F_{Bj}^{-1} F_{Ck}^{-1} F_{Dm}^{-1} \right) \dot{\mathbf{C}}_{ABCD} + \left[\mathbf{e}_{ijl} \left(\mathbf{m}^a_{ijk} - \mathbf{G}^a_{ijmn} F_{Lm}^{-1} F_{kL,n} \right) F_{Ck}^{-1} \right] \dot{W}_{lC} - \mathbf{e}_{ijl} \mathbf{G}^a_{ijkm} F_{Dm}^{-1} F_{Ck}^{-1} \dot{\mathbf{q}}_{lDC},
\end{aligned} \tag{127}$$

where, in the first equality, apexes $(\cdot)^s$ and $(\cdot)^a$ indicate the left-symmetric and the left-skew part of $(\cdot)$, respectively, while, in the second one, we used expressions (54) and (55) by applying the constraint for which $R' = R$ and $q' = q$.

The continuum is called “hyperelastic” if, along any germ of virtual motion allowed to the body, that expression coincides with the total time-derivative of a “potential” per unit mass, say φ, *i.e.* of a function of the strain characteristics C, W, $\mathbf{c}$, $\mathbf{q}$ and $\mathbf{C}$ for which

$$-\mathrm{tr}\left(\mathscr{P}^{in}\right) = \rho \left(\frac{\partial \varphi}{\partial C} \cdot \dot{C} + \frac{\partial \varphi}{\partial W} \cdot \dot{W} + \frac{\partial \varphi}{\partial \mathbf{c}} \cdot \dot{\mathbf{c}} + \frac{\partial \varphi}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial \varphi}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} \right). \tag{128}$$

An immediate implication is that the constitutive laws for the interactions are restricted to depend only on those strains (and not of the associated strainings, including H).

The class of hyperelastic continua, under the choice (115), is here very special; it may seem even incongruous as it gives strains a central role; however, that criticism is easily rebutted, if only to a degree, by a recall, already voiced cursorily earlier, of the accepted central position in physics of the perfect gas. Besides it may offer hints for efforts to understand the behaviour of some complex fluid-like materials (related matter is discussed in [3]).

Within the proviso, we obtain that:

$$\begin{aligned}
\mathrm{sym}\, \left(T_a + A_a^T + \tilde{Z}_a\right)_{ij} &= 2\rho F_{iA} F_{jB} \frac{\partial \varphi}{\partial C_{AB}} + 2\rho F_{iA} F_{jB,C} \frac{\partial \varphi}{\partial \mathbf{c}_{ABC}} - 2 \left(\rho F_{iA} F_{jB} \frac{\partial \varphi}{\partial \mathbf{C}_{ABCD}} \right)_{,CD} + \\
&\quad + 2\rho \frac{\partial \varphi}{\partial \mathbf{C}_{ABCD}} F_{El}^{-1} \left[\left(F_{iA} F_{lB,C} + F_{iB} F_{lA,C} \right) F_{jE,D} + \left(F_{iA} F_{lB} \right)_{,D} F_{jE,C} + \left(F_{iA} F_{jE} \right)_{,D} F_{lB,C} + \right. \\
&\quad \left. + 2 F_{iA} \left(F_{jE,C} F_{lB,D} - F_{jE,CD} F_{lB} \right) + F_{iA} F_{lB} F_{Gk}^{-1} F_{jE,G} F_{kC,D} \right], \\
\mathbf{m}^s_{ijk} &= 2\rho F_{iA} F_{jB} F_{kC} \frac{\partial \varphi}{\partial \mathbf{c}_{ABC}} + 2\rho \frac{\partial \varphi}{\partial \mathbf{C}_{ABCD}} \left[\left(F_{iA} F_{jB} F_{kC} \right)_{,D} + \left(F_{iA} F_{jB,C} + F_{iB} F_{jA,C} \right) F_{kD} \right], \\
\mathbf{m}_a^a &= \frac{1}{2} \rho \mathbf{e} \left[\frac{\partial \varphi}{\partial W} F^T - \frac{\partial \varphi}{\partial \mathbf{q}} (\mathrm{Grad}\; F)^T \right], \;\; \mathbf{G}^s_{ijkm} = -2\rho\, F_{iA} F_{jB} F_{kC} F_{mD} \frac{\partial \varphi}{\partial \mathbf{C}_{ABCD}}, \;\; \mathbf{G}_a^a = -\frac{1}{2} \rho\, \mathbf{e} \left[\left(\frac{\partial \varphi}{\partial \mathbf{q}} F^T \right)^t F^T \right].
\end{aligned} \tag{129}$$

Thus, if φ were known, the balance equations would become a fully declared set of partial differential equations for ρ, v, H (or, even, for ρ, x, H), when the $\tilde{Z}$ is opportunely assigned in connection with thermal concept, as in the previous §6.4; in fact, by using (123) and (129) the Cauchy equation reduces now to

$$\rho \left[\frac{\partial v}{\partial \tau} + (\mathrm{grad}\; v)\, v \right] + \mathrm{div}\, \left\{ \rho \left[H - \left(\frac{\partial L}{\partial \tau} + (\mathrm{grad}\, L)\, v \right) Y \right] \right\} = \rho f + \mathrm{div}\, \left[\hat{T} - \mathrm{div}\,({}^t\mathbf{m}_a) - \mathrm{div}^2\, \mathbf{G}_a - \tilde{Z}_a - \tilde{Z}_c \right], \tag{130}$$

where $\mathbf{m}_a$ and $\mathbf{G}_a$ are given by $(129)_{2-4}$, while the symmetric tensor $\hat{T}$ is defined by the first, second and fourth terms on the right hand side of $(129)_1$.

We examine now the consequences of the second specific choice for E:

$$Z = \mathbf{Z} \dot{M} \quad \text{and} \quad \mathbf{j} = \Gamma(\mathrm{grad}\, \dot{M}), \tag{131}$$

with $\mathbf{Z}$ symmetric also in the second two indices, as it must be Γ in the fifth and sixth one; then we have the following expression for the trace of the density of internal power (76):

$$\operatorname{tr}\left(\mathscr{P}^{in}\right) = -\Big[D \cdot \operatorname{sym} T + S^d \cdot \operatorname{skw} T + D^m \cdot \operatorname{sym} A + \mathbf{b}^s \cdot \mathbf{m}^s - \mathbf{b}^a \cdot \mathbf{m}^a + \left(\tilde{Z} + \operatorname{div}^2 \mathbf{G}\right) \cdot \dot{M} - \mathbf{G} \cdot \left(\operatorname{grad}^2 \dot{M}\right)\Big], \tag{132}$$

where we used relations (75) and (77).

Then we remark, first of all that it is possible to give a version of the power of internal actions which is linear in the time rates of strain measures, *i.e.*, C, N, Q, W, M, $\mathbf{n}$ and $\mathbf{P}$. The requisite kinematic identities have already been obtained in §3, that are relations (46), $(49)_3$, (53), (54) and (55). Substitution of L, B, $\mathbf{b}$ in (132) leads accordingly to

$$\begin{aligned} -\operatorname{tr}\left(\mathscr{P}^{in}\right) &= \frac{1}{2}\left[F^{-1}(\operatorname{sym} T)F^{-T}\right] \cdot \dot{C} + \left[R'^T\,(\operatorname{skw} T)\,R\right] \cdot \dot{Q} + \frac{1}{2}\left\{G^{-1}\left\{\operatorname{sym} A - \mathbf{m}^s\left[(\operatorname{grad} G)^t\, G^{-1}\right]^T\right\} G^{-T}\right\} \cdot \dot{N} + \\ &\quad + \left[(\mathbf{e}\,\mathbf{m}^a)F^{-1}\right] \cdot \dot{W} + \frac{1}{2}\left[G^{-1}\ {}^t\left(G^{-1}\,\mathbf{m}^s\right)F^{-T}\right] \cdot \dot{\mathbf{n}} + \left(\tilde{Z} + \operatorname{div}^2 \mathbf{G}\right) \cdot \dot{M} - \left[\left(\mathbf{G}F^{-T}\right)^t F^{-T}\right] \cdot \dot{\mathbf{P}}. \end{aligned} \tag{133}$$

In this unconstrained case, the continuum is called "hyperelastic" if, along any germ of virtual motion allowed to the body, that expression coincides with the total time-derivative of the "potential" per unit mass φ that now is a function of the strain characteristics C, N, Q, W, M, $\mathbf{n}$ and $\mathbf{P}$ for which

$$-\operatorname{tr}\left(\mathscr{P}^{in}\right) = \rho\left(\frac{\partial\varphi}{\partial C} \cdot \dot{C} + \frac{\partial\varphi}{\partial N} \cdot \dot{N} + \frac{\partial\varphi}{\partial Q} \cdot \dot{Q} + \frac{\partial\varphi}{\partial W} \cdot \dot{W} + \frac{\partial\varphi}{\partial M} \cdot \dot{M} + \frac{\partial\varphi}{\partial \mathbf{n}} \cdot \dot{\mathbf{n}} + \frac{\partial\varphi}{\partial \mathbf{P}} \cdot \dot{\mathbf{P}}\right). \tag{134}$$

An immediate implication is that the constitutive laws for the interactions are restricted to depend only on those strains (and not of the associated strainings, including H).

Under the choice (131) and positions (115) and (116), the class of hyperelastic continua has to satisfy the following constitutive prescriptions:

$$\begin{aligned} &\operatorname{sym} T = 2\rho F \frac{\partial\varphi}{\partial C} F^T, \qquad \operatorname{skw} T = \operatorname{skw} A = \rho R' \frac{\partial\varphi}{\partial Q} R^T, \qquad \operatorname{sym} A = 2\rho\, G \frac{\partial\varphi}{\partial N} G^T + \rho G \frac{\partial\varphi}{\partial \mathbf{n}} (\mathbf{n}^T G^{-1}), \\ &\mathbf{m}^s = 2\rho G\ {}^t\left[G\left(\frac{\partial\varphi}{\partial \mathbf{n}} F^T\right)\right], \qquad \mathbf{m}^a = \frac{1}{2}\rho\,\mathbf{e}\left(\frac{\partial\varphi}{\partial W} F^T\right), \\ &\tilde{Z}_{AB} = \rho \frac{\partial\varphi}{\partial M_{AB}} + \left(\rho \frac{\partial\varphi}{\partial \mathbf{P}_{ABCD}}\right)_{,CD}, \qquad \mathbf{G}_{ABkl} = -\rho \frac{\partial\varphi}{\partial \mathbf{P}_{ABCD}} F_{kC} F_{lD}. \end{aligned} \tag{135}$$

Thus, if φ were known, the balance equations would become a fully declared set of partial differential equations for ρ, v, B, H (or, even, for ρ, x, G, H).

8 Hypoelastic ephemeral continua

The title of this section could be misleading; it could even be considered a misnomer. In proposing hypo-elasticity, C. Truesdell [22] remarks that, whereas some corollaries might suggest that the theory "rests upon stress-strain relations, that is not the case ... A relation between stress and strain is thus the outcome, not the assumption, of our theory". Although he insists that his "theory really embodies no new ideas of mechanical behavior beyond those of classical linear elasticity", in our opinion he introduces in it a radically new conception: the possibility of 'mutations' embodied in the assumption that the constitutive laws may evolve; the evolution being ruled by a separate balance equation, inspired, somewhat vaguely, by those of linear elasticity. Actually, he adds, one may insert, if so desired, temperature as a further variable.

A disturbing side aspect of the theory is "the more liberal field for initial stresses, a field which was noted but not investigated"; it appears to us to be a lacuna still to be given physical motivation.

Our pretense is to exalt aspects mentioned almost as digressions and invite the reader to interpret, within this section, our system (16)-(20) as the combination of four strictly mechanical equations plus a last one which introduces mutations via a relatively foreign variable H, which could, like temperature, be viewed as belonging, strictly, to a different chapter of physics. As mentioned repeatedly, we interpret H as a function, the initial values of which can (in fact must) be naturally be assigned as part of known circumstances. Thus we transfer the responsibility of mutation to H rather than invoking (or inventing) a direct evolution law for T.

Truesdell proposal refers to a continuum without meso-phenomena apart from those caused by the agitation of molecules embodied, within our approach, in the tensor H. So, to come close to the proposal, we must again introduce the constraint $G=F$ or, expressed via rates, $B=L$. However, we must also abide by Truesdell constitutive rule for the compulsion leading to mutation, a rule which we attribute to Z; a surface compulsion, typified by $\mathbf{j}$, being absent. Accepting the transfer formally (see (1.6) from [22]) we write

$$Z=\mathbf{Z}D \quad \text{and} \quad \mathbf{j}\equiv\mathbf{0}. \tag{136}$$

But now, to follow Truesdell again, we admit the compulsion $\mathbf{Z}$ to depend on strainings: H and possibly D and we must decide if the macro/meso-scopic constraint would breed additive reactions on molecular agitation; on this matter we take a number of decisions the wisdom of which must be decided by the appropriateness of consequences. We decompose $\mathbf{Z}$, trivially, as the sum

$$\mathbf{Z}=\left(\mathbf{Z}-\frac{2}{3}I\otimes\tilde{Z}\right)+\frac{2}{3}I\otimes\tilde{Z}, \tag{137}$$

with $\tilde{Z}$ given by (75); only the latter enters the power (74), consequently we presume that the macro/meso-scopic constraint breeds only reactions $\tilde{Z}_r$. The question arises now if all those decisions may be compatible with the usual characterization of perfect constraints, tentatively

$$(T_r+A_r^T)\cdot L+\tilde{Z}_r\cdot D=0, \tag{138}$$

where B is forced to coincide with L, while the arbitrariness of L and H persists. Then, similarly to a predicament met already, (138) is generally non-linear in strainings, linearity being assured only by indifference of $\mathbf{Z}$ to changes in D or H.

But such neglect would be a slight to the suggestions of Truesdell. Rather, we repeat and stretch a proposal already advanced for $\tilde{Z}_r$ in §6.2, even summoning the impact of microstructural variables beyond the mesoscopic ones, either by extending our theory beyond the present setting or affirming that such variables are only latent here and embodied in terms of mesoscopic ones.

An example of the former suggestion is the splitting of the number density of molecules into ranges of specific kinetic energy as predicated in introductions to thermodynamics.

To offer a specific example of the latter, we recall and adapt the conception from [4]. The microstructural variables could, for further generality, be even coordinates of points of a manifold, but then they are interpreted as functions of strains; that being the case, they are said to become 'latent'. It may be implied that the original structural generality is surely downgraded; but that need not be the case: the global dimension of the strains one can call upon may assure the fit. Think of a case mimicking nematic liquid crystals where the manifold is the unit sphere with identification of antipodes; that manifold may be embedded in the space of strains C, occupying there the surface of the tensor product of two identical unit vectors.

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