DOI: 10.1478/AAPP.952A6

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 95, No. 2, A6 (2017)

PURSUIT-EVASION GAME OF MANY PLAYERS WITH COORDINATE-WISE INTEGRAL CONSTRAINTS ON A CONVEX SET IN THE PLANE

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ABSTRACT. We study a differential game of many pursuers and one evader in the plane. It is assumed that the pursuers and evader move is allowed within a non empty closed convex set in the plane. Control functions of players are subject to coordinate-wise integral constraints. The game is over when the state of the evader y coincides with that of a pursuer x_i, $i = \{1,...,m\}$ at given time t_i (unspecified), *i.e.*, $x_i(t_i) = y(t_i)$. We obtain conditions under which the game is over in finite time, no matter where the players start from. Moreover, we construct winning for the pursuers.

1. Introduction

In game theory, differential games are those games which could be modeled by a dynamical system. In a pursuit-evasion game, one or more pursuers try to capture one or more evaders that try to avoid capture. Among others, Isaacs Isaacs (1965) and Petrosyan Petrosyan (1977) have studied differential games and pursuit-evasion problems.

Satimov *et al.* (1983) studied a linear pursuit differential game of many pursuers and one evader in the space $\mathbb{R}^n$ with integral constraints on controls of players. In his paper, the objects move according to the equations

$$\dot{z}_i = C_i z_i - u_i + v, \quad z_i(t_0) = z_i^0, \quad i = 1,\dots,m$$

where u_i and v are the control parameters of the i-th pursuer and evader, respectively. In the main result of this research it is assumed the eigenvalues of the matrices C_i are real. It is shown that if the total resources of controls of the pursuers is greater than that of the evaders, then under certain conditions pursuit can be completed.

The differential game of one pursuer and one evader with integral constraints, studied by Ibragimov (2002), occurs on a closed convex subset S of $\mathbb{R}^n$ with the dynamics of the players described by

$$\dot{x} = \alpha(t)u, \quad x(0) = x_0, \quad \int_0^\infty |u(s)|^2 ds \le \rho^2$$

$$\dot{y} = \alpha(t)v, \quad y(0) = y_0, \quad \int_0^\infty |v(s)|^2 ds \le \sigma^2.$$

He also gives a formula for optimal pursuit time and optimal strategies of the players. Ibragimov (2004) gives a pursuit differential game of m pursuers and k evaders with integral constraints described by the systems of differential equations

$$\dot{z}_{ij} = C_{ij}z_i + u_i - v_j, \quad z_{ij}(t_0) = z_{ij}^0, \quad i = 1,...,m \quad j = 1,...,k,$$

$$\int_0^\infty |u_i(s)|^2 ds \le \rho_i^2 \;\; i = 1,...,m; \quad \int_0^\infty |v_j(s)|^2 ds \le \sigma_j^2, \quad j = 1,...,k$$

where u_i is the control parameter of the i-th pursuer, and v_j is that of the j-th evader. Here, the eigenvalues of matrices C_{ij} are not necessarily real, and moreover, the number of evaders is arbitrary (but finite). Under assumption that the total resource of controls of the pursuers is greater than that of the evaders, that is

$$\rho_1^2 + \rho_2^2 + ... + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + ... + \sigma_k^2,$$

and the real parts of all eigenvalues of the matrices C_{ij} are nonpositive, it is proved that pursuit can be completed from any initial position.

Ibragimov and Salimi (2009) investigated a differential game for inertial players with integral constraints under the assumption that the control resource of the evader is less than that of each pursuer. Ibragimov *et al.* (2012) investigated an evasion from many pursuers in simple motion differential games with integral constraints. Recently, Salimi *et al.* (2016) studied a differential game in which countably many dynamical objects pursue a single one. All the players perform simple motions and the control of a group of pursuers are subject to integral constraints whereas the control of the other pursuers and the evader are subject to geometric constraints. The authors construct optimal strategies for players and find the value of the game. Salimi (2017) studied an evasion differential game with finite number of pursuers and one evader in Hilbert space. The control functions of players are subject to the geometric constraints. He solves the game by presenting an strategy for the evader which guarantees its evasion.

In the present paper, we consider a differential game of several pursuers and one evader with coordinate-wise integral constraints. The game occurs in a nonempty closed convex set in the plane. We obtain sufficient conditions for the completion of the game and give winning strategies.

2. Statement of the problem

We consider a differential game described by the equations

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad i = 1,...m, \quad \dot{y} = v, \quad y(0) = y_0, \tag{1}$$

where x_i, u_i, y, $v \in \mathbb{R}^2$, u_i is the control parameter of the pursuer x_i and v is that of the evader y.

Definition 2.1. *A measurable function $u_i(t) = (u_{i1}(t), u_{i2}(t))$; $t \ge 0$, is called admissible control of the pursuer x_i if*

$$\int_0^\infty |u_{ij}(s)|^2 ds \le \rho_{ij}^2, \tag{2}$$

where ρ_{ij}, for $i = 1,...,m,\ j = 1,2$, are given positive numbers.

Definition 2.2. *A measurable function $v(t) = (v_1(t), v_2(t))$; $t \geq 0$, is called admissible control of the evader if*

$$\int_0^\infty |v_j(s)|^2 ds \leq \sigma_j^2, \tag{3}$$

where σ_j, for $j = 1,2$ are given positive numbers.

Definition 2.3. *A Borel measurable function $U_i(x_i, y, v) = (U_{i1}(x_i, y, v),$ $U_{i2}(x_i, y, v))$, $U_i : \mathbb{R}^6 \to \mathbb{R}^2$, is called a strategy of the pursuer x_i if for any control of the evader $v(t)$, $t \geq 0$, the initial value problem*

$$\dot{x}_i = U_i(x_i, y, v(t)),\ \ x_i(0) = x_{i0}, \quad \dot{y} = v(t),\ \ y(0) = y_0,$$

has a unique solution $(x_i(t), y(t))$ and the inequalities

$$\int_0^\infty |U_{i1}(x_i(s), y(s), v(s))|^2 ds \leq \rho_{i1}^2, \quad \int_0^\infty |U_{i2}(x_i(s), y(s), v(s))|^2 ds \leq \rho_{i2}^2$$

hold.

Definition 2.4. *We say that the pursuit can be completed from the initial position $\{x_{10}, ...,$ $x_{m0}, y_0\}$ at time T in the game (1) with conditions (2) and (3), if there exist strategies U_i, $i = 1,...,m$, of the pursuers such that for any control $v = v(\cdot)$ of the evader, the equality $x_i(t) = y(t)$ holds for some $i \in \{1,...,k\}$ and $t \in [0,T]$.*

It is assumed that the players could not leave a given nonempty compact and closed convex set $N \subset \mathbb{R}^n$.

3. Main result

Now we formulate the main result of the paper.

Theorem 3.1. *If the inequality*

$$\rho_{1j}^2 + \rho_{2j}^2 + ... + \rho_{mj}^2 > \sigma_j^2 \tag{4}$$

holds for some $j \in \{1,2\}$ then the pursuit can be completed at a finite time T in the game (1), with conditions (2) and (3) from any initial position.

Proof. We prove the theorem when the equality (4) holds for $j = 1$. Thus,

$$\rho_{11}^2 + \rho_{21}^2 + ... + \rho_{m1}^2 > \sigma_1^2$$

and $\rho_{i2},\ i = 1,2,...,m$, are positive numbers. Put

$$\sigma_{i1} = \frac{\sigma_1}{\rho_1}\rho_{i1},\ i = 1,2,...,m;\ \ \rho_1 = (\rho_{11}^2 + \rho_{21}^2 + ... + \rho_{m1}^2)^{1/2}.$$

Clearly, $\sigma_{i1} < \rho_{i1}$. Let x_0 and y_0 are points in N such that $|x_0 - y_0| = diam\, N := \max_{x,y \in N}|x - y|$. We pass the x-axis through the points x_0 and y_0. Let $d = |x_0 - y_0|$, $c = \max_{(\xi,\eta)\in N}|\eta|$. Without loss of generality, we may assume that all pursuers at the beginning are on the x-axis, since otherwise we can move them to that position by applying the following controls:

$$u_1(t) = 0,\ \ u_2(t) = -\frac{x_{i2}^0}{T},\ \ 0 \le t \le T;\ \ T = \max_{i \in \{1,\dots,m\}} \frac{4|x_{i2}^0|^2}{\rho_{i2}^2}, \tag{5}$$

and consider the game with $\rho_{i2}' = (\sqrt{3}/2)\rho_{i2}$ instead of ρ_{i2}.

The strategy of each pursuer is constructed in two stages. In the first stage, the pursuer moves with constant speed along x-axis to get one vertical line with the evader (upper side or lower side). In the second stage, the pursuer uses a strategy we soon describe.

Let us construct a strategy for the pursuer x_i, $i = 1,\dots,m$, on an interval $[\theta_i, \theta_{i+1}]$; with $\theta_1 = 0$. We define θ_{i+1} inductively below. Set

$$\begin{aligned} &u_{i1}(t) = sgn(y_1(\theta_i) - x_{i1}(\theta_i))\frac{d}{t_{i1}}, \\ &u_{i2}(t) = 0, \quad \theta_i < t \le \theta_i + \tau_{i1}, \quad i = 1,\dots,m, \end{aligned} \tag{6}$$

where τ_i, $0 \le \tau_i \le t_{i1}$, is the time for which $x_{i1}(\theta_i + \tau_{i1}) = y_1(\theta_i + \tau_{i1})$, $t_{i1} = \frac{d^2}{\rho_{i1}^2 - \sigma_{i1}^2}$. Clearly, such a time τ_{i1} exists since

$$\left| \int_{\theta_i}^{\theta_i + t_{i1}} u_{i1}(t)dt \right| = t_{i1} \cdot \frac{d}{t_{i1}} = d,$$

that is, on the time interval $[\theta_i, \theta_i + t_{i1}]$, the pursuer x_i can travel a distance d.

We now construct the second part of the strategy of the pursuer x_i. Set

$$\begin{aligned} &u_{i1}(t) = v_1(t), \quad u_{i2}(t) = sgn(y_2(\theta_i + \tau_{i1}) - x_{i2}(\theta_i + \tau_{i1}))\frac{c}{t_{i2}}, \\ &\theta_i + \tau_i < t \le \theta_{i+1}, \quad \text{if} \quad x_i(t) \in int\ N, \\ &\text{and} \\ &u_{i1}(t) = 0,\ \ u_{i2}(t) = 0, \quad \text{if} \quad x_i(t) \in \partial N, \end{aligned} \tag{7}$$

where ∂N and $int\ N$ are the boundary and interior of N, $t_{i2} = \frac{c^2}{\rho_{i2}^2}$, and $\theta_{i+1} = \theta_i + \tau_{i1} + t_{i2}$. On the time interval $[\theta_i, \theta_{i+1}]$ all the other pursuers x_j, $j \in \{1,\dots,i-1,i+1,\dots m\}$ do not move. In other words,

$$u_j(t) \equiv 0,\ \ j \in \{1,\dots,i-1,i+1,\dots m\},\ \ t \in [\theta_i, \theta_{i+1}].$$

Note that

$$x_{i1}(t) = y_1(t),\ \ \theta_i + \tau_{i1} \le t \le \theta_{i+1},$$

whenever

$$\int_{\theta_i + \tau_{i1}}^{\theta_{i+1}} v_1^2(t)dt \le \sigma_{i1}^2, \tag{8}$$

since under this condition we have

$$\begin{aligned}\int_{\theta_i}^{\theta_{i+1}} u_{i1}^2(t)dt &= \int_{\theta_i}^{\theta_i+\tau_{i1}} u_{i1}^2(t)dt + \int_{\theta_i+\tau_{i1}}^{\theta_{i+1}} u_{i1}^2(t)dt \\ &= \frac{a^2}{t_{i1}} + \int_{\theta_i+\tau_{i1}}^{\theta_{i+1}} v_1^2(t)dt \\ &\le \rho_{i1}^2 - \sigma_{i1}^2 + \sigma_{i1}^2 = \rho_{i1}^2.\end{aligned}$$

Therefore, if (8) holds, then the pursuer x_i is able to apply the strategy (7) that ensures the equality

$$x_{i2}(\theta_i + \tau_{i1} + \tau_{i2}) = y_2(\theta_i + \tau_{i1} + \tau_{i2}),$$

for some $0 \le \tau_{i2} \le t_{i2}$. If (8) fails for a control of the evader $v(t) = (v_1(t), v_2(t))$, then the pursuit may not be completed by the pursuer x_i. However, (8) holds at least for one index $i \in \{1, ..., m\}$. Indeed, if

$$\int_{\theta_i+\tau_{i1}}^{\theta_{i+1}} v_1^2(t)dt > \sigma_{i1}^2$$

for all $i \in \{1, ..., m\}$, then we have

$$\int_0^{\theta_{m+1}} v_1^2(t)dt = \sum_{i=1}^m \int_{\theta_i}^{\theta_{i+1}} v_1^2(t)dt \ge \sum_{i=1}^m \int_{\theta_i+\tau_{i1}}^{\theta_{i+1}} v_1^2(t)dt > \sum_{i=1}^m \sigma_{i1}^2 = \sigma_1^2,$$

which is a contradiction. Thus the pursuit could be completed by a pursuer x_i. □

4. Discussion and conclusion

We have obtained a sufficient condition (4) to complete the pursuit in the differential game of m pursuers and one evader with coordinate-wise integral constraints. We have constructed strategies of the pursuers and showed that pursuit can be completed from any initial position in N.

Let us discuss information which is used by the pursuer. To bring positions of all pursuers to x-axis by the control (5), the pursuers use only their initial positions. Next, the pursuer x_i using $y_1(\theta_i)$ and $x_{i1}(\theta_i)$ constructs $u_i(t)$ at the time θ_i by formula (6) and uses it on $(\theta_i, \theta_i + \tau_{i1}]$. While using (6) the pursuer x_i observes whether $x_{i1}(t) = y_1(t)$. This is possible since by Definition **2.3**, the pursuer x_i is allowed to know $x_{i1}(t)$ and $y_1(t)$. Finally, as soon as $x_{i1}(t) = y_1(t)$ at some time $t = \theta_i + \tau_{i1}$, the pursuer x_i uses (7), which requires knowing $v_1(t)$. By Definition **2.3**, the pursuer x_i knows $v(t)$, in particular, knows $v_1(t)$ as well.

As a concrete example, let $m = 2$, $\rho_{11} = 1$, $\rho_{21} = 1.1$, $\sigma_1 = \sqrt{2}$ and ρ_{12}, ρ_{22} be any positive numbers and

$$N = \left\{ (x,y) \,\middle|\, \frac{x^2}{9} + \frac{y^2}{4} \le 1 \right\}.$$

Then one can see that $d = 6$, and $c = \max\{|y|\ ;\ (x,y) \in N\} = 2$ and by Theorem **3.1** pursuit can be completed.

Acknowledgments

This research was supported by the Research Grant of the Universiti Putra Malaysia, No. 05-02-12-1868RU, and by MEDAlics, the Research Centre at the University Dante Alighieri (Reggio Calabria, Italy).

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Manuscript received 13 June 2017; communicated 19 June 2017; published online 2 November 2017

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