EXISTENCE RESULTS FOR A NEUMANN PROBLEM INVOLVING THE p(x)-LAPLACIAN WITH DISCONTINUOUS NONLINEARITIES

GIUSEPPINA BARLETTA, ANTONIA CHINNÌ, AND DONAL O'REGAN

Abstract. In this paper the existence of a nontrivial solution to a parametric Neumann problem for a class of nonlinear elliptic equations involving the $p(x)$-Laplacian and a discontinuous nonlinear term, is established. Under a suitable condition on the behavior of the potential at 0^+, we obtain an interval $]0, \lambda^*]$, such that, for any $\lambda \in]0, \lambda^*]$ our problem admits at least one nontrivial weak solution. The solution is obtained as a critical point of a locally Lipschitz functional. In addition to providing a new conclusion on the existence of a solution even for $\lambda = \lambda^*$, our theorem also includes other results in the literature for regular problems.

1. Introduction

In this paper we are interested in the existence of non-trivial weak solutions of the Neumann problem

$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) \text{ in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \qquad (N_\lambda)$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$, $p \in C(\bar{\Omega})$, $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplace operator, a is a function in $L^\infty(\Omega)$, λ is a positive parameter and ν is the outward unit normal to $\partial\Omega$.

The $p(x)$-Laplacian operator arises in generalized Lebesgue and Sobolev spaces (see for example [26], [18], [35]). Contributions to nonlinear elliptic problems associated with the $p(x)$-Laplacian from various view points can be found in a large number of papers in the literature. The main result of this paper, Theorem 3.1, establishes the existence of one non-trivial weak solution for problem (N_λ) when the nonlinear term $f = f(x,t)$ is almost everywhere continuous with respect the variable t. We note that equations involving discontinuous nonlinear terms arise frequently in free boundary problems and obstacle problems. Chang in [12] extended variational methods to a class of non-differentiable functionals and later S.A. Marano and D. Motreanu in [27] and [28] obtained a non-smooth version of B. Ricceri's three critical-points theorem (see [32], [33]). To study problem (N_λ) via a variational approach, we use an abstract nonsmooth critical point result established in [9] in which a recent critical point result of Bonanno (see [3]) has been extended to the nonsmooth framework.

Hereafter we suppose that $p \in C(\bar{\Omega})$ verifies

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty, \qquad (1)$$

while for $f : \Omega \times \mathbb{R} \to \mathbb{R}$ we require that

(f_1) f is measurable with respect to each variable separately;

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Corresponding author: Antonia Chinnì, Tel. +39 090 3977324; E-mail address: achinni@unime.it.

(f_2) there exist $a_1, a_2 \in [0, +\infty[$ and $q \in C(\bar{\Omega})$ with $1 < q(x) < p^*(x)$ for each $x \in \bar{\Omega}$, such that

$$|f(x,t)| \leq a_1 + a_2|t|^{q(x)-1}$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases} \tag{2}$$

Our result, Theorem 3.1, extends Theorem 3.1 of [2] where the existence of one non trivial weak solution for problem (N_λ) was established under continuity assumptions on the nonlinear term f. We note as in [2], that we remove the assumption $p^- > N$ which appears in most papers in the literature involving the $p(x)$-Laplacian (see for example [7], [8], [11], [13], [14], [20], [22], [21], [24], [25], [25], [29], [30], [19]). In the growth condition (f_2) there is no relation between functions q and p except that $q(x) < p^*(x)$ for each $x \in \bar{\Omega}$. Finally, we observe that Theorem 3.1 provides a precise estimate of the interval of parameters λ where a trivial weak solution exists. We now present a simple consequence of the main result.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function satisfying*

$(\tilde{f}_2)$ $|f(t)| \leq a_1 + a_2|t|^{q-1}$
for each $t \in \mathbb{R}$, for some $a_1, a_2 \in [0, +\infty[$, $1 < q < (p^*)^-$ and $\max\{a_1, a_2\} < \frac{1}{\bar{k}}$ where $\bar{k} := a_1 k_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q} (k_q)^q (p^+)^{\frac{q}{p^-}}$;

$(\tilde{f}_4)$ $\limsup_{t\to 0^+} \dfrac{\int_0^t f(\xi)\, d\xi}{t^{p^-}} = +\infty.$

Moreover, put $D_f := \{z \in \mathbb{R} : f(\cdot) \text{ is discontinuous at } z\}$, and we suppose that

$(\tilde{f}_3)$ *for each $z \in D_f$*

$$f^-(z) := \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{\ |\xi - z| < \delta} f(\xi) > [|z|]^{p-2} := \max\{|z|^{p^- - 2}, |z|^{p^+ - 2}\}.$$

Then the problem

$$\begin{cases} -\Delta_{p(x)} u + |u|^{p(x)-2}u = f(u) \text{ in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{$\tilde{N}_\lambda$}$$

admits at least one non trivial weak solution.

The paper is arranged as follows. In Section 2 we present some auxiliary results that we use in Section 3 to prove our main theorem. In Section 4 we give some examples of functions satisfying the assumptions in our main result.

2. Preliminaries

Here and in the sequel, we suppose that $p \in C(\bar{\Omega})$ satisfies condition (1). The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx < +\infty\}\cdot$$

On $L^{p(x)}(\Omega)$ we consider the norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}. \tag{3}$$

With such norms, $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces. If we assume that

(H_1) *$a \in L^\infty(\Omega)$, with $a_- := \mathrm{ess\,inf}_\Omega a(x) > 0$,*

then on $W^{1,p(x)}(\Omega)$ it is possible to consider the following norm

$$\|u\|_a = \inf\left\{\sigma > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\sigma}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\sigma}\right|^{p(x)}\right) dx \leq 1\right\},$$

which is equivalent to that introduced in (3). In particular (see [2])

$$\frac{[a_-]_{\frac{1}{p}}}{1 + [a_-]_{\frac{1}{p}}}\|u\|_{W^{1,p(x)}(\Omega)} \leq \|u\|_a \leq (1 + \|a\|_\infty)^{\frac{1}{p^-}} \|u\|_{W^{1,p(x)}(\Omega)} \tag{4}$$

for each $u \in W^{1,p(x)}(\Omega)$, where, for $\alpha > 0$ and $h \in C(\bar\Omega)$ with $1 < h^-$, we put

$$[\alpha]^h := \max\{\alpha^{h^-}, \alpha^{h^+}\}$$

and

$$[\alpha]_h := \min\{\alpha^{h^-}, \alpha^{h^+}\}.$$

The following result generalizes the well-known Sobolev embedding theorem.

Theorem 2.1. ([22, Proposition 2.5]) *Assume that $p \in C(\bar\Omega)$ with $p(x) > 1$ for each $x \in \bar\Omega$. If $r \in C(\bar\Omega)$ and $1 < r(x) < p^*(x)$ for all $x \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ where p^* is the critical exponent related to p defined in (2).*

In the sequel, we will denote by k_r the best constant for which one has

$$\|u\|_{L^{r(x)}(\Omega)} \leq k_r\|u\|_a \quad \text{for all } u \in W^{1,p(x)}(\Omega)\,. \tag{5}$$

We define the functional $\Phi : W^{1,p(x)}(\Omega) \to \mathbb{R}$ as

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right) dx \quad \text{for all } u \in W^{1,p(x)}(\Omega). \tag{6}$$

Proposition 2.1. [11, Proposition 2.2]

Let $u \in W^{1,p(x)}(\Omega)$.

(j) If $\|u\|_a < 1$ then $\dfrac{1}{p^+}\|u\|_a^{p^+} \leq \Phi(u) \leq \dfrac{1}{p^-}\|u\|_a^{p^-}$.

(jj) If $\|u\|_a > 1$ then $\dfrac{1}{p^+}\|u\|_a^{p^-} \leq \Phi(u) \leq \dfrac{1}{p^-}\|u\|_a^{p^+}$.

Standard arguments (see for example [29] and [20]) imply that Φ is sequentially weakly lower semi-continuous and that it is a C^1 functional on $W^{1,p(x)}(\Omega)$, with derivative

$$\langle \Phi'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv \right) dx, \tag{7}$$

for any $u, v \in W^{1,p(x)}(\Omega)$. Moreover (see [11, Lemma 3.1]), Φ' is an homeomorphism.

Given any Banach space X, by X^* we denote the topological dual of X, and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). Let $(X, \|\cdot\|)$ be a real Banach space. A functional $h : X \to \mathbb{R}$ is said to be locally Lipschitz when, to each $x \in X$, there corresponds a neighbourhood U of x and a constant $L \geq 0$, depending on x, such that

$$|h(y) - h(z)| \leq L\|y - z\|$$

for each $y, z \in U$. For a locally Lipschitz function $h : X \to \mathbb{R}$, the generalized directional derivative of h at point $x \in X$, in the direction $v \in X$, is defined as

$$h^0(x; v) := \limsup_{u \to x; t \to 0^+} \frac{h(u + tv) - h(u)}{t},$$

while the generalized gradient of h in x is the set

$$\partial h(x) := \{w^* \in X^* : \langle w^*, v\rangle \leq h^0(x; v) \ \forall v \in X\}\cdot$$

Basic properties of generalized directional derivative and generalized gradient can be found in [15] and in [12].

If $h : X \to \mathbb{R}$ is a locally Lipschitz functional and $x \in X$, then we say that x is a critical point of h if it satisfies the inequality

$$h^0(x; y) \geq 0$$

for all $y \in X$ or, equivalently, $0 \in \partial h(x)$.

Definition 2.1. *If $h : X \to \mathbb{R}$ is a locally Lipschitz functional that can be written as $h = \Phi - \Upsilon$, with Φ and Υ locally Lipschitz, and $r \in \mathbb{R}$, we say that h verifies the Palais-Smale condition cut off upper at r ($(PS)^{[r]}$-condition for short) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that*

(α) *$\{h(u_n)\}_{n\in\mathbb{N}}$ is bounded;*

(β) *there exists a sequence $\{\epsilon_n\}_{n\in N}$ in $]0, +\infty[$, $\epsilon_n \to 0^+$ such that*

$$h^0(u_n; v) \geq -\epsilon_n\|v\| \quad \text{for all } v \in X$$

(γ) *$\Phi(u_n) < r$ for all $n \in \mathbb{N}$;*

has a convergent subsequence.

The energy functional associated to (N_λ) can be written as

$$I_\lambda = \Phi - \lambda\Psi$$

where Φ is that in (6) and $\Psi : W^{1,p(x)}(\Omega) \to \mathbb{R}$ is defined as

$$\Psi(u) = \int_\Omega F(x, u(x))dx \qquad \text{for all } u \in W^{1,p(x)}(\Omega), \tag{8}$$

with $F(x,\xi) := \int_0^\xi f(x,t)\, dt$ for each $(x,\xi) \in \Omega \times \mathbb{R}$. If f satisfies the growth condition (f_2) then Proposition 2.2 of [5] ensures that the functional Ψ is locally Lipschitz on $W^{1,p(x)}(\Omega)$ and so I_λ is locally Lipschitz rather than smooth.

We recall that, for fixed $\lambda > 0$, a point $u \in W^{1,p(x)}(\Omega)$ is a weak solution to (N_λ) if

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv)dx = \lambda \int_\Omega f(x, u(x))v(x)dx \tag{9}$$

holds for each $v \in W^{1,p(x)}(\Omega)$.

In Section 3, Lemma 3.1, we will show that any critical point of I_λ satisfies (9) and is hence a solution to (N_λ). From this lemma, it thus suffices to show that I_λ has a non trivial critical point and to do this we shall use the next result.

Theorem 2.2. ([9], Theorem 2.5) *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functionals such that $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that:*

(a_1) $\dfrac{\sup\limits_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(a_2) *for each* $\lambda \in \left] \dfrac{\Phi(\bar{x})}{\Psi(\bar{x})}, \dfrac{r}{\sup\limits_{\Phi(x)\le r} \Psi(x)} \right[$ *the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$ condition.*

Then, for each $\lambda \in \Lambda_r := \left] \dfrac{\Phi(\bar{x})}{\Psi(\bar{x})}, \dfrac{r}{\sup\limits_{\Phi(x)\le r} \Psi(x)} \right[$, *there exists a critical point of I_λ, $x_{0,\lambda} \in \Phi^{-1}(]0, r[)$ such that $I_\lambda(x_{0,\lambda}) \le I_\lambda(x)$ for all $x \in \Phi^{-1}(]0, r[)$.*

We conclude this section by specifying the type of discontinuity allowed on the nonlinear term f in (N_λ). We denote by $\mathcal{G}$ the family of highly discontinuous functions, that is the family of all locally bounded functions $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(m_1) $g(\cdot, z)$ is measurable for every $z \in \mathbb{R}$;

(m_2) there exists a set $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 0$ such that the set

$$D_g := \bigcup_{x\in\Omega\setminus\Omega_0} \{z \in \mathbb{R} : g(x, \cdot) \text{ is discontinuous at } z\}$$

has measure zero;

(m_3) the functions

$$g^-(x, z) := \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|\xi - z|<\delta} g(x, \xi), \quad g^+(x, z) := \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|\xi - z|<\delta} g(x, \xi)$$

are superpositionally measurable i.e. $g^-(\cdot, u(\cdot))$ and $g^+(\cdot, u(\cdot))$ are measurable provided $u : \Omega \to \mathbb{R}$ is measurable too.

Clearly, if $f \in \mathcal{G}$ then f satisfies (f_1).

3. Main result

A first result obtained from Theorem 2.2 is the existence of one nontrivial solution for problem (N_λ) when the function f is discontinuous with respect to the second variable.

Theorem 3.1. *Let $f \in \mathcal{G}$, satisfy (f_2),*

(f_3) *for each $\lambda > 0$, for a.e. $x \in \Omega$ and each $z \in D_f$ the condition $\lambda f^-(x, z) \le a(x)|z|^{p(x)-2}z \le \lambda f^+(x, z)$ implies $\lambda f(x, z) = a(x)|z|^{p(x)-2}z$,*

(f_4)

$$\limsup_{t\to 0^+} \frac{\int_\Omega F(x, t)\, dx}{t^{p^-}} = +\infty.$$

Then, with $\lambda^* = \dfrac{1}{a_1 k_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-} [k_q]^q (p^+)^{\frac{q^+}{p^-}}}$, *where k_1 and k_q are given by (5), for each $\lambda \in]0, \lambda^*]$ problem (N_λ) admits at least one non trivial weak solution.*

The proof of Theorem 3.1 needs two preliminary lemmas.

Lemma 3.1. *If $f \in \mathcal{G}$, satisfies (f_2) and (f_3) then for each $\lambda > 0$, the critical points of the functional I_λ are weak solutions for problem (N_λ).*

Proof. Put $X := W^{1,p(x)}(\Omega)$. Fixed $\lambda > 0$, if $u_0 \in X$ is a critical point of $\Phi - \lambda\Psi$, then one has

$$(\Phi - \lambda\Psi)^0(u_0; v) \geq 0 \quad \text{for all } v \in X. \tag{10}$$

Since $\Phi \in C^1(X)$, we have, in particular

$$0 \leq (\Phi - \lambda\Psi)^0(u_0; v) \leq \Phi^0(u_0; v) + (-\lambda\Psi)^0(u_0; v) = \langle \Phi'(u_0), v\rangle + (-\lambda\Psi)^0(u_0; v),$$

whence

$$-\langle \Phi'(u_0), v\rangle \leq (-\lambda\Psi)^0(u_0; v) \quad \text{for all } v \in X\,.$$

This means

$$-\Phi'(u_0) \in \partial(-\lambda\Psi)(u_0),$$

that is (see [15])

$$\Phi'(u_0) \in \partial(\lambda\Psi)(u_0). \tag{11}$$

Now, from Theorem 1.8 of [23], X is embedded and dense in $L^{q(x)}(\Omega)$ and so, from Theorem 2.2 of [12] one has

$$\partial(-\lambda\Psi)(u_0) \subseteq \partial\left(-\lambda\Psi_{|L^{q(x)}(\Omega)}\right)(u_0). \tag{12}$$

From (f_2) and because $-\lambda f \in \mathcal{G}$ too, we deduce that f^+, f^-, $-\lambda f^+$ and $-\lambda f^-$ satisfy all the assumptions in Theorem 2.1 of [12]. Thus

$$\partial(\lambda\Psi)_{L^{q(x)}(\Omega)}(u_0) \subseteq \lambda[f^-(x, u_0(x)), f^+(x, u_0(x))]_{q'}$$

where (see [12]) q' is the conjugate function of q, i.e. $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ and

$$[f^-(\cdot, u_0(\cdot)), f^+(\cdot, u_0(\cdot))]_{q'} =$$

$$\{W \in L^{q'(x)}(\Omega) : W(x) \in [f^-(x, u_0(x)), f^+(x, u_0(x))] \text{ a.e. in } \Omega\}.$$

Using (11),

$$\Phi'(u_0) \in \lambda[f^-(\cdot, u_0(\cdot)), f^+(\cdot, u_0(\cdot))]_{q'}\,.$$

and then by (7) we have

$$-\Delta_{p(x)}u \in [-a(\cdot)|u_0(\cdot)|^{p(x)-2}u_0(\cdot) + \lambda f^-(\cdot, u_0(\cdot)), -a(x)|u_0(\cdot)|^{p(x)-2}u_0(\cdot) + \lambda f^+(\cdot, u_0(\cdot))]_{q'}\,.$$

Thus, there exists a unique

$$W_0 \in [-a(\cdot)|u_0(\cdot)|^{p(x)-2}u_0(\cdot) + \lambda f^-(\cdot, u_0(\cdot)), -a(\cdot)|u_0(\cdot)|^{p(x)-2}u_0(\cdot) + \lambda f^+(\cdot, u_0(\cdot))]_{q'}$$

such that

$$\int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v dx = \int_\Omega W_0 v dx \tag{13}$$

for each $v \in L^{q(x)}(\Omega)$. Now, to conclude, we show that

$$W_0(x) = -a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f(x, u_0(x)) \quad \text{a.e. in } \Omega\,. \tag{14}$$

Let $\Omega_1 \subseteq \Omega$ be a set of measure zero such that

$$W_0(x) \in [-a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f^-(x, u_0(x)), -a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f^+(x, u_0(x))] \tag{15}$$

for all $x \in \Omega \setminus \Omega_1$. Put

$$\Omega_f := \{x \in \Omega : \, u_0(x) \in D_f\}$$

and note that $\Omega_f = u_0^{-1}(D_f)$. From (13), u_0 is a weak solution of the problem

$$\begin{cases} -\Delta_{p(x)} u = W_0(x) \text{ in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{N_0}$$

Now using Lemma 1 of [17] and arguing as in [5] (Proposition 2.4), we obtain $W_0(x) = 0$ a.e. on Ω_f. Let $\Omega_3 \subseteq \Omega_f$ be a set of measure zero such that

$$W_0(x) = 0 \tag{16}$$

for all $x \in \Omega_f \setminus \Omega_3$. From (f_3), there exists a set $\Omega_2 \subseteq \Omega$ of measure zero such that for all $x \in \Omega \setminus \Omega_2$ and each $z \in D_f$ the condition $\lambda f^-(x, z) \le a(x)|z|^{p(x)-2}z \le \lambda f^+(x, z)$ implies

$$\lambda f(x, z) = a(x)|z|^{p(x)-2}z. \tag{17}$$

Put $\Omega^* = \cup_{i=0}^{3} \Omega_i$, and we want to prove that

$$W_0(x) = -a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f(x, u_0(x)$$

for each $x \in \Omega \setminus \Omega^*$. Clearly $m(\Omega^*) = 0$. Take $x \in \Omega \setminus \Omega^*$. If $u_0(x) \notin D_f$, taking into account that in particular $x \in \Omega \setminus \Omega_0$ (see (m_2)), the definition of D_f leads to continuity at x of the function $f(\cdot, u_0(\cdot))$. From (15) one has

$$W_0(x) = -a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f(x, u_0(x).$$

If $u_0(x) \in D_f$, since $x \in (\Omega \setminus \Omega_1) \cap (\Omega \setminus \Omega_3)$ then

$$0 = W_0(x) \in [-a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f^-(x, u_0(x)), -a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f^+(x, u_0(x))]$$

i.e.

$$\lambda f^-(x, u_0(x)) \le a(x)|u_0(x)|^{p(x)-2}u_0(x) \le \lambda f^+(x, u_0(x)).$$

Also, $x \in \Omega \setminus \Omega_2$ and $u_0(x) \in D_f$, and so from (17) one has

$$\lambda f(x, u_0(x)) = a(x)|u_0(x)|^{p(x)-2}u_0(x)$$

and so

$$W_0(x) = 0 = -a(x)|u_0(x)|^{p(x)-2}u_0(x) + \lambda f(x, u_0(x)).$$

We have proved (14) and the conclusion follows from (13) and (14). □

Lemma 3.2. *Assume that $p \in C(\overline{\Omega})$ satisfies* (1) *and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying (f_1) and (f_2). Then, for each $\lambda > 0$, the functional I_λ satisfies the Palais-Smale condition $(PS)^{[r]}$, for any $r > 0$.*

Proof. Fixed $\lambda > 0$, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W^{1,p(x)}(\Omega)$, satisfying (α), (β) and (γ) in Definition 2.1. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded and we can extract a subsequence (again call it $\{u_n\}$), converging to $u \in W^{1,p(x)}(\Omega)$, weakly in $W^{1,p(x)}(\Omega)$ and strongly in $L^{r(x)}(\Omega)$, for any function r as in Theorem 2.1. From [15] we know that $(-\lambda\Psi)^0$ is u.s.c. and

$$\left(-\lambda\Psi_{|W^{1,p(x)}(\Omega)}\right)^0(u; v) \le (-\lambda\Psi)^0(u; v) \text{ for all } u, v \in W^{1,p(x)}(\Omega),$$

so

$$\limsup_{n\to+\infty}\left(-\lambda\Psi_{|W^{1,p(x)}(\Omega)}\right)^{0}(u_n;u_n-u)\leq\limsup_{n\to+\infty}(-\lambda\Psi)^{0}(u_n;u_n-u)=0\,. \tag{18}$$

If we choose $v=u_n-u$ in the condition (β) of Definition 2.1 ($I^0_\lambda(u_n;v)\geq-\epsilon_n\|v\|$) and pass to the upper limit, then

$$\limsup_{n\to+\infty}\langle\Phi^{'}(u_n),u-u_n\rangle\leq\limsup_{n\to+\infty}\left(-\lambda\Psi_{|W^{1,p(x)}(\Omega)}\right)^{0}(u_n;u_n-u)\leq 0\,.$$

Thus (see [20]) $u_n\to u$ in $W^{1,p(x)}(\Omega)$. □

Proof of Theorem 3.1 Put $X:=W^{1,p(x)}(\Omega)$ equipped by norm $\|\cdot\|_a$, and we consider the functional

$$I_\lambda(\cdot):=\Phi(\cdot)-\lambda\Psi(\cdot)$$

introduced in the previous section. As before, Φ and Ψ satisfy the regularity assumptions required in Theorem 2.2. In particular Lemma 3.2 ensures that the condition (a_2) is satisfied for all $r,\lambda>0$ while Lemma 3.1 ensures that for each $\lambda>0$, the critical points of the functional I_λ are weak solutions for problem (N_λ).

We first show the existence of a nontrivial solution for $\lambda\in]0,\lambda^*[$. Fix $\lambda\in]0,\lambda^*[$, and choose $r=1$ to satisfy condition (a_1) of Theorem 2.2. From (f_4) there exists

$$0<\xi_\lambda<\min\left\{1,\left(\frac{p^-}{\|a\|_\infty|\Omega|}\right)^{\frac{1}{p^-}}\right\} \tag{19}$$

such that

$$\frac{p^-\int_\Omega F(x,\xi_\lambda)\,dx}{\xi_\lambda^{p^-}\|a\|_\infty|\Omega|}>\frac{1}{\lambda}\cdot \tag{20}$$

We denote by $\bar u$ the function of X defined by

$$\bar u(x)=\xi_\lambda$$

for each $x\in\Omega$ and we observe that

$$\Phi(\bar u)\leq\frac{1}{p^-}\|a\|_\infty|\Omega|[\xi_\lambda]^p<1 \tag{21}$$

and

$$\Psi(\bar u)=\int_\Omega F(x,\xi_\lambda)\,dx.$$

Condition (f_2) implies

$$|F(x,t)|\leq a_1|t|+\frac{a_2}{q(x)}|t|^{q(x)}$$

for each $(x,t)\in\Omega\times\mathbb{R}$. For $u\in\Phi^{-1}(]-\infty,1])$ we have

$$\Psi(u)\leq a_1\int_\Omega|u(x)|\,dx+\frac{a_2}{q^-}\int_\Omega|u(x)|^{q(x)}\,dx=a_1\|u\|_{L^1(\Omega)}+\frac{a_2}{q^-}\rho_q(u)\cdot$$

Theorem 1.3 of [23] and the embeddings $W^{1,p(x)}(\Omega)\hookrightarrow L^1(\Omega)$ and $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ ensure that

$$a_1\|u\|_{L^1(\Omega)}+\frac{a_2}{q^-}\rho_q(u)\leq a_1\|u\|_{L^1(\Omega)}+\frac{a_2}{q^-}\left[\|u\|_{L^{q(x)}(\Omega)}\right]^q\leq a_1k_1\|u\|_a+\frac{a_2}{q^-}\left[k_q\|u\|_a\right]^q. \tag{22}$$

Taking into account that for each $u \in \Phi^{-1}(]-\infty, 1])$, from Proposition 2.1, one has

$$\|u\|_a \leq (p^+)^{\frac{1}{p^-}},$$

and thus (20) and (22) lead to

$$\sup_{\Phi(u)\leq 1} \Psi(u) \leq a_1 k_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-} [k_q]^q (p^+)^{\frac{q^+}{p^-}} = \frac{1}{\lambda^*} < \frac{1}{\lambda} < \frac{p^- \int_\Omega F(x,\xi_\lambda)\, dx}{\xi_\lambda^{p^-} \|a\|_\infty |\Omega|} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}, \tag{23}$$

and so condition (a_1) of Theorem 2.2 is satisfied. Since $\lambda \in \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \lambda^* \right[\subseteq \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{1}{\sup_{\Phi(u)\leq 1}\Psi(u)} \right[$, Theorem 2.2 guarantees the existence of a local minimum point u_λ for the functional I_λ such that

$$0 < \Phi(u_\lambda) < 1,$$

and so u_λ is a non-trivial weak solution of problem (N_λ).

Now we prove that (N_{λ^*}) has a nontrivial solution. We have a sequence $\{u_\lambda\}$, bounded in $W^{1,p(x)}(\Omega)$ satisfying $I_\lambda(u_\lambda) \leq I_\lambda(v)$ for all $v \in \Phi^{-1}(]0,1[)$.
Thus, taking into account (j) of Proposition 2.1,

$$I_\lambda(u_\lambda) \leq I_\lambda(v_n) \tag{24}$$

for any sequence $\{v_n\} \subseteq \Phi^{-1}(]0,1[)$, such that $\|v_n\|_a \to 0$.

Then, from the continuity of I_λ we have

$$I_\lambda(u_\lambda) \leq 0 \quad \text{for every } \lambda \in]0, \lambda^*[\,. \tag{25}$$

Take now $\bar{\lambda} \in]0,\lambda^*[$ and a sequence $\{\lambda_n\}_{n\in N}$, such that $0 < \bar{\lambda} < \lambda_n < \lambda^*$ for any $n \in N$ and $\lambda_n \nearrow \lambda^*$. Then $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is bounded and we can extract a subsequence (again call it $\{u_n\}$ for simplicity), converging to $u^* \in W^{1,p(x)}(\Omega)$, weakly in $W^{1,p(x)}(\Omega)$ and strongly in $L^{r(x)}(\Omega)$, for any function r as in Theorem 2.1. From (9),

$$\langle \Phi'(u_n), u^* - u_n \rangle = \lambda_n \int_\Omega f(x, u_n(x))(u^* - u_n)dx$$

for all $n \in N$. Thus

$$\limsup_{n\to+\infty} \langle \Phi'(u_n), u^* - u_n \rangle = \limsup_{n\to+\infty} \lambda_n \int_\Omega f(x, u_n(x))(u^* - u_n)dx = 0\,, \tag{26}$$

and (see [20]) $u_n \to u^*$ in $W^{1,p(x)}(\Omega)$. Passing to the limit in

$$\langle \Phi'(u_n), v \rangle = \lambda_n \int_\Omega f(x, u_n(x))v(x)dx$$

we obtain

$$\langle \Phi'(u^*), v \rangle = \lambda^* \int_\Omega f(x, u^*(x))v(x)dx \quad \text{for every } v \in W^{1,p(x)}(\Omega), \tag{27}$$

and thus u^* is a solution to (N_λ), with $\lambda = \lambda^*$. Now we show that $u^* \neq 0$. We know that every u_λ is a minimun point for I_λ and that $I_\lambda(u_\lambda) \leq I_\lambda(v)$ for all $v \in \Phi^{-1}(]0,1[)$ and all $\lambda \in]0,\lambda^*[$. Thus, since $u_\lambda \in \Phi^{-1}(]0,1[)$ for every $\lambda \in]0,\lambda^*[$,

$$I_{\lambda_n}(u_n) \leq I_{\lambda_n}(u_{\bar{\lambda}}) \quad \text{and} \quad I_{\bar{\lambda}}(u_{\bar{\lambda}}) \leq I_{\bar{\lambda}}(u_n) \quad \text{for every } n \in N\,. \tag{28}$$

Adding the previous inequalities and considering that $\bar{\lambda} < \lambda_n$ for every $n \in N$ we obtain

$$\Psi(u_{\bar{\lambda}}) \leq \Psi(u_n) \text{ for every } n \in N\,,$$

whence

$$\Psi(u_{\bar{\lambda}}) \leq \Psi(u^*)\,. \tag{29}$$

Now, arguing by contradiction, if $u^* = 0$ then $\Psi(u_{\bar{\lambda}}) \leq 0$, hence $I_{\bar{\lambda}}(u_{\bar{\lambda}}) > 0$, but this contradicts (25). □

Remark 3.1. A careful reading of Lemma 3.1 shows that if we remove hypothesis (f_3) then Theorem 3.1 guarantees the existence of a nontrivial solution to the differential inclusion

$$\begin{cases} -\Delta_{p(x)} u + a(x)|u|^{p(x)-2}u \in \lambda[f^-(x,u), f^+(x,u)] \text{ a.e. in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \end{cases} \qquad (\tilde{N}_\lambda)$$

Any solution to $(\tilde{N}_\lambda)$ turns into a solution to (N_λ) only for those particular values of λ for which condition (f_3) applies.

Corollary 3.1. *Let $f \in \mathcal{G}$, satisfy $f(x,0) = 0$ a.e. in Ω, (f_2), (f_4) and*

(f_3') for each $\lambda > 0$, for a.e. $x \in \Omega$ and each $z \in D_f \cap]0, +\infty[$ the condition $\lambda f^-(x,z) \leq a(x)|z|^{p(x)-2}z \leq \lambda f^+(x,z)$ implies $\lambda f(x,z) = a(x)|z|^{p(x)-2}z$.

Then, with $\lambda^* = \dfrac{1}{a_1 k_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-}\,[k_q]^q\,(p^+)^{\frac{q^+}{p^-}}}$, where k_1 and k_q are given by (5), for each $\lambda \in]0, \lambda^*]$ problem (N_λ) admits at least one non trivial non negative weak solution.

Proof. We truncate f through the function f_+ as follows:

$$f_+(x,t) = \begin{cases} 0 & \text{if } t < 0,\ x \in \Omega \\ f(x,t) & \text{if } t \geq 0\ x \in \Omega. \end{cases}$$

In a standard way, define $F_+(x,t) = \int_0^t f_+(x,\tau)d\tau$ and $\Psi_+(u) = \int_\Omega F_+(x, u(x))dx$. We want to apply Theorem 3.1 to the functional $I_{+,\lambda}(\cdot) = \Phi(\cdot) - \lambda\Psi_+(\cdot)$. Clearly $I_{+,\lambda}$ satisfies all the assumptions of Theorem 3.1. For the sake of completeness we point out only that $D_{f_+} = D_f \cap]0, +\infty[$ and so f_+ satisfies (f_3). Theorem 3.1 guarantees the existence of a local minimum point u_λ for the functional $I_{+,\lambda}$ such that

$$0 < \Phi(u_\lambda) < 1$$

and

$$\langle \Phi'(u_\lambda), v\rangle = \lambda \int_\Omega f_+(x, u_\lambda(x))v(x)dx \text{ for every } v \in W^{1,p(x)}(\Omega). \qquad (30)$$

Acting on (30) with $v = \max\{-u_\lambda, 0\}$ and arguing as in [1], Theorem 3.1, we obtain $u_{+,\lambda} \geq 0$ a.e. in Ω. Thus (30) is equivalent to (9) and $u_{+,\lambda}$ is a non trivial non negative weak solution of (N_λ). □

Proof of Theorem 1.1 We apply Theorem 3.1 by choosing $a(x) = 1$ for each $x \in \Omega$. Clearly $f \in \mathcal{G}$ and conditions $(\tilde{f}_2)$ and $(\tilde{f}_4)$ establish conditions (f_2) and (f_4) respectively. Moreover, due to the choice of a_1 and a_2, $1 \in]0, \lambda^*]$. Now, we verify that, for a.e. $x \in \Omega$ and each $z \in D_f$, condition

$$f^-(z) \leq |z|^{p(x)-2}z \leq f^+(z) \qquad (31)$$

implies

$$\lambda f(z) = |z|^{p(x)-2}z.$$

Clearly, if $z \in D_f \cap]-\infty, 0]$ then (31) doesn't hold. If $z \in D_f \cap]0, +\infty[$ then from condition $(\tilde{f_3})$ we have

$$f^-(z) > z^{p(x)-1}$$

which conflicts with (31) which implies $f^-(z) \leq z^{p(x)-1}$. Thus condition (f_3) of Theorem 3.1 holds for $\lambda = 1$. Taking into account Remark 3.1, the conclusion of Theorem 1.1 is obtained. □

4. Examples

Now, we present some examples of functions verifying the assumptions in Theorem 3.1. We explicitly note that all the functions below fail to satisfy the continuity assumptions required in [2].

Example 4.1. Take a function $a \in L^\infty(\Omega)$ satisfying $a_- > 0$ and consider the function $f : \Omega \times R \to R$ defined as

$$f(x,t) = \begin{cases} \frac{\sin(t-h\pi)}{t-h\pi} & \text{if } x \in \Omega \text{ and } t \in]h\pi, (h+1)\pi] \text{ for some } h \leq 0\,. \\ g(x,t) & \text{if } x \in \Omega \text{ and } t > \pi, \end{cases} \tag{32}$$

where $g : \Omega \times]\pi, +\infty[$ is continuous and satisfies the following conditions

- there exist $a_1, a_2 \in [0, +\infty[$ and $q \in C(\bar{\Omega})$ with $1 < q(x) < p^*(x)$ for each $x \in \bar{\Omega}$, such that
$$|g(x,t)| \leq a_1 + a_2 |t|^{q(x)-1}$$
for each $(x,t) \in \Omega \times]\pi, +\infty[$;
- $\lim_{t\to\pi^+} g(x,t) = l(x) \leq 0$ a.e. in Ω.

We observe that $f \in \mathcal{G}$. In fact if we choose $\Omega_0 = \emptyset$ then $D_f = \{k\pi : k \leq 1\}$, $m(D_f) = 0$. Further

$$f^-(x,k\pi) = f(x,k\pi) = 0,\ f^+(x,k\pi) = 1, \text{for } k \leq 0, \tag{33}$$
$$f^-(x,\pi) = l(x),\ f^+(x,\pi) = f(x,\pi) = 0\,.$$

Now, take $\lambda \in]0, \lambda^*]$, where λ^* is defined as in Theorem 3.1, $x \in \Omega$ and $t \in D_f$. Then, taking into account (33), the inequality (f_3) reads as

$$0 \leq a(x)|k\pi|^{p(x)-2}k\pi \leq \lambda \text{ for some } k \leq 0, \tag{34}$$

or

$$\lambda l(x) \leq a(x)\pi^{p(x)-1} \leq 0 \text{ for } k = 1\,. \tag{35}$$

We observe that condition (34) implies $k = 0$. Indeed, for $k < 0$ the inequality cannot occur because the function in the middle is negative, while if $k = 0$ then we have $f(x, 0) = 0$ and (f_3) holds. For $k = 1$ the inequality in (f_3) turns into (35) and this cannot happen as well. Thus (f_3) is satisfied. For $(x,t) \in \Omega \times]0, \pi[$ we have

$$F(x,t) = \int_0^t \frac{\sin(\tau)}{\tau} d\tau\,. \tag{36}$$

Using De L'Hopital's rule and the fact that $p^- > 1$, we obtain

$$\limsup_{t\to 0^+} \frac{\int_\Omega F(x,t)dx}{t^{p_-}} = \limsup_{t\to 0^+} \frac{|\Omega| \int_0^t \frac{\sin\tau}{\tau} d\tau}{t^{p_-}} = \limsup_{t\to 0^+} |\Omega| \frac{\sin t}{p_- t^{p_-}} = +\infty\,, \tag{37}$$

namely (f_4).

Example 4.2. Assume $p_- > 2$ and consider a function $a \in L^\infty(\Omega)$ satisfying

$$a_- \geq \frac{1}{k_1(p_+)^{\frac{1}{p_-}}}\,. \tag{38}$$

Then, for each $\lambda \in]0, \frac{1}{k_1(p_+)^{\frac{1}{p_-}}}]$, the problem

$$\begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda(u - [u]) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{39}$$

admits at least one non trivial non negative weak solution. We consider the function $f : R \to R$ by

$$f(t) = t - [t] \text{ where } [t] \text{ is the integer part of } t \in R\,. \tag{40}$$

Since $f(0) = 0$, we apply Corollary 3.1. Clearly f satisfies (f_1) and (f_2) with $a_1 = 1$, $a_2 = 0$. Further, $\lim_{t\to k^+} f(t) = 0 = f(k)$ and $\lim_{t\to k^-} f(t) = 1$ for $k \in Z$. We can choose $\Omega_0 = \emptyset$, while $D_f = Z$ and $|D_f| = 0$. For $t \in (0, 1)$ we have

$$F(t) = \frac{t^2}{2}\,, \tag{41}$$

and

$$\limsup_{t\to 0^+} \frac{\int_\Omega F(t)dx}{t^{p_-}} = +\infty\,, \tag{42}$$

namely (f_4). With our choice of f, (f_3') reads as

> for each $\lambda > 0$, and each $k > 0$ the condition $0 \leq a(x)|k|^{p(x)-2}k \leq \lambda$ implies $0 = a(x)|k|^{p(x)-2}k$.

We observe that, if it where true, then we have

$$a_- \leq a(x)k^{p(x)-1} \leq \lambda \leq \frac{1}{k_1(p_+)^{\frac{1}{p_-}}} \tag{43}$$

and this contradicts (38). The conclusion follows by Corollary 3.1 taking into account that $\lambda^* = \frac{1}{k_1(p_+)^{\frac{1}{p_-}}}$.

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References

[1] Barletta G., *Existence results for semilinear elliptical hemivariational inequalities*, Nonlinear Anal., **68**, no. 8 (2008), 24172430.

[2] Barletta G. and Chinnì A., *Existence of solutions for a Neumann problem involving the $p(x)$-Laplacian*, Electronic Journal of Differential Equations, **2013**, n.158 (2013), 1–12.

[3] Bonanno G., *A critical point theorem via the Ekeland variational principle*, Nonlinear Analysis, **75** (2012), 2992–3007.

[4] Bonanno G. and Candito P., *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations, **244** (2008), 3031–3059.

[5] Bonanno G. and Chinnì A., *Discontinuous elliptic problems involving the $p(x)$-Laplacian*, Math. Nachr., **284**, n.5,6 (2011), 639–652.

[6] Bonanno G. and Chinnì A., *Multiple solutions for elliptic problems involving the p(x)-Laplacian*, Le Matematiche, **66** (2011), 105–113.

[7] Bonanno G. and Chinnì A., *Existence results of infinitely many solutions for p(x)-Laplacian elliptic Dirichlet problems*, Complex Variables and Elliptic Equations,**57**, n. 11 (2012), 1233-246.

[8] Bonanno G. and Chinnì A., *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent*, J. Math. Anal. Appl.,**418**, (2014), 812827.

[9] Bonanno G., D'Aguì G. and Winkert P. *Sturm-Liouville equations involving discontinuous nonlinearities*, Minimax Theory and its Applications, **01** (2015) n.1, to appear.

[10] Bonanno G. and Marano S. A., *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal., **89** (2010), 1–10.

[11] Cammaroto F., Chinnì A. and Di Bella B., *Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian*, Nonlinear Anal., **71** (2009), 4486–4492.

[12] Chang K.-C. F., *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[13] Chao Ji, *Remarks on the existence of three solutions for the p(x)-Laplacian equations*, Nonlinear Anal., **74** (2011), 2908–2915.

[14] Chinnì A. and Livrea R. *Multiple solutions for a Neumann-type differential inclusion problem involving the $p(\cdot)$-Laplacian*, Discrete and Continuous Dynamical Systems(Series S), **5 (4)** (2012), 753–764.

[15] Clarke F. H. , *A new approach to Lagrange multipliers*, Math. Oper. Res. **1** (1976), 165–174.

[16] Clarke F. H. , *Optimization and Nonsmooth Analysis*, Classics Appl. Math. SIAM **5** (1990).

[17] De Giorgi E., Buttazzo G. and Dal Maso G., *On the lower semicontinuity of certain integral functionals*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl., **74** (1983), 274–282.

[18] Diening L., Harjulehto P., Hästö P. and Ružička M., *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, **2017**, Springer-Verlag, Heidelberg (2011).

[19] Ding X. and Shi X., *Existence and multiplicity of solutions for a general p(x)-Laplacian Neumann problem*, Nonlinear Analysis, **70** (2009), 3715-3720.

[20] Fan X. L. and Deng S. G., *Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations*, Nonlinear Anal., **67** (2007), 3064–3075.

[21] Fan X. L. and Ji C., *Existence of infinitely many solutions for a Neumann problem involving the p(x)-Laplacian*, J. Math. Anal. Appl., **334** (2007), 248–260.

[22] Fan X. L. and Zhang Q.H., *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal., **52** (2003), 1843–1852.

[23] Fan X. L. and Zhao D., *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., **263** (2001), 424–446.

[24] Liu D., *Existence of multiple solutions for a $p(x)$-Laplace equations*, Electron. J. Differential Equations, Vol. 2010(2010) **33**, 1-11.

[25] Liu Q., *Existence of three solutions for $p(x)$-Laplacian equations*, Nonlinear Anal., **68** (2008), 2119–2127.

[26] Kováčik O. and Rákosník J., *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math., **41** (1991), 592–618.

[27] Marano S. A. and Motreanu D., *On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems*, Nonlinear Anal. **48** (2002), 37–52.

[28] Marano S. A. and Motreanu D., *Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian*, J. Differential Equations **182** (2002), 10820

[29] Mihăilescu M., *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Nonlinear Anal., **67** (2007), 1419–1425.

[30] Moschetto D. S., *A quasilinear Neumann problem involving the $p(x)$-Laplacian*, Nonlinear Analysis, **71** (2009), 2739–2743.

[31] Musielak J., *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034**, Springer, Berlin (1983).

[32] Ricceri B., *On a three critical points theorem*, Arch. Math. **75** (2000), 220–226.

[33] Ricceri B., *A general variational principle and some of its applications*, J. Comput. Appl. Math. **113** (2000), 401410.

[34] Ružička M., *Electrorheological fluids: modeling and mathematical theory,* Springer-Verlag, Berlin 2000.

[35] Cruz-Uribe D.V. and Fiorenza A., *Variable Lebesgue Spaces: Foundations and Harmonic Analysis,* Birkhauser, 2013.

(G. Barletta) Università degli Studi Mediterranea di Reggio Calabria, DICEAM-Dipartimento di Ingegneria Civile e Ambientale, Via Graziella, Località Feo di Vito, 89100 Reggio Calabria, Italy.
E-mail address: giuseppina.barletta@unirc.it

(A. Chinnì) Department of Civil, Information Technology, Construction, Environmental Engineering and Applied Mathematics, University of Messina, 98166- Messina,Italy
E-mail address: achinni@unime.it

(D. O'Regan) School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie